Strategic Games: Nash Equilibria and Social Optima

Krzysztof R. Apt

(so not Krzystof and definitely not Krystof)

CWI, Amsterdam, the Netherlands,
University of Amsterdam
Basic Concepts
Best response.
Nash equilibrium.
Examples.
Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set $S_i$ of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$, for each player $i$.

Notation: $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others’ rationality.
### Three Examples

#### Prisoner’s Dilemma

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<tr>
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#### The Battle of the Sexes

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#### Matching Pennies

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Main Concepts

- **Notation:** \( s_i, s'_i \in S_i, s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n. \)

- \( s_i \) is a **best response** to \( s_{-i} \) if
  \[
  \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).
  \]

- \( s \) is a **Nash equilibrium** if \( \forall i \) \( s_i \) is a best response to \( s_{-i} \):
  \[
  \forall i \in \{1, \ldots, n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).
  \]

- **Intuition:** In a Nash equilibrium no player can gain by **unilaterally** switching to another strategy.
Nash Equilibrium

Prisoner’s Dilemma: 1 Nash equilibrium

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The Battle of the Sexes: 2 Nash equilibria

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Matching Pennies: no Nash equilibrium

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Traveler’s dilemma

- 2 players,
- Strategies of each player: \(\{2, \ldots, 100\}\),
- Payoff functions:

\[
p_i(s) := \begin{cases} 
  s_i & \text{if } s_i = s_{-i} \\
  s_i + 2 & \text{if } s_i < s_{-i} \\
  s_{-i} - 2 & \text{otherwise}
\end{cases}
\]

\((2, 2)\) is a unique Nash equilibrium.
Example: The 2nd Maldives Mr & Miss Beauty Contest.
Beauty-contest Game

[Moulin, ’86]

- each set of strategies = \{1, \ldots, 100\},
- payoff to each player:
  1 is split equally between the players whose submitted number is closest to \(\frac{2}{3}\) of the average.

Example
submissions: 29, 32, 29; average: 30,
payoffs: \(\frac{1}{2}, 0, \frac{1}{2}\).

(1, \ldots, 1) is a Nash equilibrium.
Pareto efficient outcomes.
Social welfare.
Social optima.
Examples.
s is Pareto efficient if for no \( s' \)

\[
\forall i \in \{1, \ldots, n\} \quad p_i(s') \geq p_i(s), \\
\exists i \in \{1, \ldots, n\} \quad p_i(s') > p_i(s).
\]

Social welfare of \( s \): \( \sum_{j=1}^{n} p_j(s) \).

s is a social optimum if \( \sum_{j=1}^{n} p_j(s) \) is maximal.
Prisoner’s Dilemma for $n$ Players

- $n > 1$ players,
- two strategies: 
  1 (formerly $C$),
  0 (formerly $D$).

$$p_i(s) := \begin{cases} 
2 \sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\
2 \sum_{j \neq i} s_j & \text{if } s_i = 1 
\end{cases}$$

- For $n = 2$ we get the original Prisoner’s Dilemma game.
- $\sum_{j \neq i} s_j$ equals the number of 1 strategies in $s_{-i}$.
- Let $\mathbf{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$.
- $\mathbf{0}$ is the unique Nash equilibrium, with social welfare $n$.
- Social optimum: $\mathbf{1}$, with social welfare $2n(n - 1)$. 
Common resources: goods that are not excludable (people cannot be prevented from using them) but are rival (one person’s use of them diminishes another person’s enjoyment of it).

Examples: congested toll-free roads, fish in the ocean, the environment, . . . ,

Problem: Overuse of such common resources leads to their destruction.

This phenomenon is called the tragedy of the commons (Hardin ’81).
Tragedy of the Commons I

(Gardner ’95)

- \( n > 1 \) players,
- two strategies:
  1 (use the resource),
  0 (don’t use),
- payoff function:

\[
p_i(s) := \begin{cases} 
0.1 & \text{if } s_i = 0 \\ 
F(m)/m & \text{otherwise}
\end{cases}
\]

where \( m = \sum_{j=1}^{n} s_j \) and

\[
F(m) := 1.1m - 0.1m^2.
\]
Tragedy of the Commons I, ctd

- payoff function:

\[ p_i(s) := \begin{cases} 
0.1 & \text{if } s_i = 0 \\
F(m)/m & \text{otherwise}
\end{cases} \]

where \( m = \sum_{j=1}^{n} s_j \) and \( F(m) := 1.1m - 0.1m^2 \).

- Note: \( F(m)/m \) is strictly decreasing,
  \( F(9)/9 = 0.2, F(10)/10 = 0.1, F(11)/11 = 0. \)

- Nash equilibria:
  \( n < 10 \): all players use the resource,
  \( n \geq 10 \): 9 or 10 players use the resource,

- Social optimum: 5 players use the resource.
Tragedy of the Commons II

(Osborne ’04)

- $n > 1$ players,
- strategies: $[0, 1]$,
- payoff function:

$$p_i(s) := \begin{cases} 
  s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\
  0 & \text{otherwise}
\end{cases}$$
payoff function:

\[ p_i(s) := \begin{cases} 
  s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\
  0 & \text{otherwise}
\end{cases} \]

‘Best’ Nash equilibrium:
when each \( s_i = \frac{1}{n+1} \),
with social welfare \( \frac{n}{(n+1)^2} \) and \( \sum_{j=1}^{n} s_j = \frac{n}{n+1} \).

Social optimum, when \( \sum_{j=1}^{n} s_j = \frac{1}{2} \),
with social welfare \( \frac{1}{4} \).

For all \( n > 1 \), \( \frac{n}{(n+1)^2} < \frac{1}{4} \).

\( \lim_{n \to \infty} \frac{n}{(n+1)^2} = 0 \) and \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \).
Cournot Competition I

(Cournot, 1838)

- One infinitely divisible product (oil),
- \( n \) companies decide **simultaneously** how much to produce,
- price is decreasing in total output.

We assume that for each player \( i \):
- his strategy set is \( \mathbb{R}_+ \),
- his payoff function is defined by

\[
p_i(s) := s_i(a - b \sum_{j=1}^{n} s_j) - cs_i
\]

for some given \( a, b, c \), where \( a > c \) and \( b > 0 \).
Cournot Competition II

- payoff function:

\[ p_i(s) := s_i(a - b \sum_{j=1}^{n} s_j) - cs_i \]

- Unique Nash equilibrium: when each

\[ s_i = \frac{a - c}{(n + 1)b} \]

- Price of the product in Nash equilibrium:

\[ a - b \sum_{j=1}^{n} s_j = a - b \frac{n(a - c)}{(n + 1)b} = \frac{a + nc}{n + 1} \]
Price of the product in Nash equilibrium:

\[
\frac{a + nc}{n + 1}.
\]

Social optimum, when \( \sum_{j=1}^{n} s_j = \frac{a-c}{2b} \).

Price of the product in a social optimum:

\[
a - b \sum_{j=1}^{n} s_j = a - b \frac{a-c}{2b} = \frac{a + c}{2}
\]

But \( a > c \) implies

\[
\frac{a + c}{2} > \frac{a + nc}{n + 1}.
\]

So the competition (more firms) drives the price down.