Strategic Games: Nash Equilibria and Social Optima

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Basic Concepts

Overview

- Best response.
- Nash equilibrium.
- Examples.

Strategic Games: Definition

Strategic game for $n \ge 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i: S_1 \times \ldots \times S_n \to \mathbb{R}$,

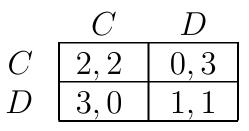
for each player *i*. Notation: $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

Basic assumptions:

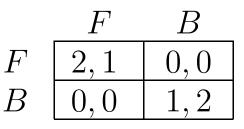
- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples

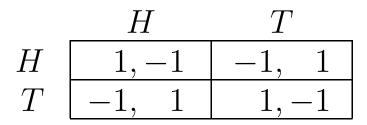
Prisoner's Dilemma



The Battle of the Sexes



Matching Pennies



Main Concepts

- Notation: $s_i, s'_i \in S_i, s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s_i is a best response to s_{-i} if

$$\forall s_i' \in S_i \ p_i(s_i, s_{-i}) \ge p_i(s_i', s_{-i}).$$

■ s is a Nash equilibrium if $\forall i \ s_i$ is a best response to s_{-i} :

$$\forall i \in \{1, ..., n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \ge p_i(s'_i, s_{-i}).$$

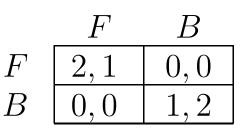
Intuition: In a Nash equilibrium no player can gain by unilaterally switching to another strategy.

Nash Equilibrium

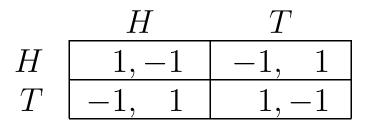
Prisoner's Dilemma: 1 Nash equilibrium

	C	D
C	2, 2	0,3
D	3,0	1, 1

The Battle of the Sexes: 2 Nash equilibria



Matching Pennies: no Nash equlibrium



Traveler's dilemma

- 2 players,
- Strategies of each player: $\{2, \ldots, 100\}$,
- Payoff functions:

$$p_i(s) := \begin{cases} s_i & \text{if } s_i = s_{-i} \\ s_i + 2 & \text{if } s_i < s_{-i} \\ s_{-i} - 2 & \text{otherwise} \end{cases}$$

(2,2) is a unique Nash equilibrium.

Beauty-contest Game

Example: The 2nd Maldives Mr & Miss Beauty Contest.



Beauty-contest Game

[Moulin, '86]

- each set of strategies = $\{1, \ldots, 100\}$,
- payoff to each player:

1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.

Example

submissions: 29, 32, 29; average: 30, payoffs: $\frac{1}{2}$, 0, $\frac{1}{2}$.

 $(1, \ldots, 1)$ is a Nash equilibrium.

Overview

- Pareto efficient outcomes.
- Social welfare.
- Social optima.
- Examples.

Definitions

 \checkmark s is Pareto efficient if for no s'

$$\forall i \in \{1, ..., n\} \ p_i(s') \ge p_i(s),$$

 $\exists i \in \{1, ..., n\} \ p_i(s') > p_i(s).$

- **Social welfare of** $s: \sum_{j=1}^{n} p_j(s)$.
- *s* is a social optimum if $\sum_{j=1}^{n} p_j(s)$ is maximal.

Prisoner's Dilemma for n **Players**

- n > 1 players,
- two strategies:
 1 (formerly C),
 0 (formerly D).

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0\\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For n = 2 we get the original Prisoner's Dilemma game.
- $\sum_{j \neq i} s_j$ equals the number of 1 strategies in s_{-i} .
- **Let** $\mathbf{1} = (1, ..., 1)$ and $\mathbf{0} = (0, ..., 0)$.
- **•** Is the unique Nash equilibrium, with social welfare n.
- Social optimum: 1, with social welfare 2n(n-1).

Tragedy of the Commons

- Common resources: goods that are are not excludable (people cannot be prevented from using them) but are rival (one person's use of them diminishes another person's enjoyment of it).
- Examples: congested toll-free roads, fish in the ocean, the environment, ...,
- Problem: Overuse of such common resources leads to their destruction.
- This phenomenon is called the tragedy of the commons (Hardin '81).

Tragedy of the Commons I

(Gardner '95)

- n > 1 players,
- two strategies:
 1 (use the resource),
 0 (don't use),
- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0\\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$

Tragedy of the Commons I, ctd

payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0\\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and $F(m) := 1.1m - 0.1m^2$.

- Note: F(m)/m is strictly decreasing, F(9)/9 = 0.2, F(10)/10 = 0.1, F(11)/11 = 0.
- Solution Nash equilibria: n < 10: all players use the resource, n ≥ 10: 9 or 10 players use the resource,
- Social optimum: 5 players use the resource.

Tragedy of the Commons II

(Osborne '04)

- n > 1 players,
- strategies: [0,1],
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^n s_j) & \text{if } \sum_{j=1}^n s_j \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Tragedy of the Commons II, ctd

payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^n s_j) & \text{if } \sum_{j=1}^n s_j \le 1\\ 0 & \text{otherwise} \end{cases}$$

- 'Best' Nash equilibrium: when each $s_i = \frac{1}{n+1}$, with social welfare $\frac{n}{(n+1)^2}$ and $\sum_{j=1}^n s_j = \frac{n}{n+1}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{1}{2}$, with social welfare $\frac{1}{4}$.

• For all
$$n > 1$$
, $\frac{n}{(n+1)^2} < \frac{1}{4}$.

•
$$\lim_{n \to \infty} \frac{n}{(n+1)^2} = 0$$
 and $\lim_{n \to \infty} \frac{n}{n+1} = 1$.

Cournot Competition I

(Cournot, 1838)

- One infinitely divisible product (oil),
- \bullet n companies decide simultaneously how much to produce,
- price is decreasing in total output.

We assume that for each player *i*:

- In this strategy set is \mathbb{R}_+ ,
- his payoff function is defined by

$$p_i(s) := s_i(a - b \sum_{j=1}^n s_j) - cs_i$$

for some given a, b, c, where a > c and b > 0.

Cournot Competition II

payoff function:

$$p_i(s) := s_i(a - b\sum_{j=1}^n s_j) - cs_i$$

Unique Nash equilibrium: when each

$$s_i = \frac{a-c}{(n+1)b}.$$

Price of the product in Nash equilibrium:

$$a - b \sum_{j=1}^{n} s_j = a - b \frac{n(a-c)}{(n+1)b} = \frac{a+nc}{n+1}.$$

Cournot Competition II, ctd

Price of the product in Nash equilibrium:

 $\frac{a+nc}{n+1}.$

- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$.
- Price of the product in a social optimum:

$$a - b\sum_{j=1}^{n} s_j = a - b\frac{a-c}{2b} = \frac{a+c}{2}$$

• But a > c implies

$$\frac{a+c}{2} > \frac{a+nc}{n+1}.$$

So the competition (more firms) drives the price down.