Chapter 1

Nash Equilibrium

Assume a set $\{1, \ldots, n\}$ of players, where $n > 1$. A ***strategic game*** (or ***non-cooperative game***) for n players, written as $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, consists of

- a non-empty (possibly infinite) set S_i of ***strategies***,
- a ***payoff function*** $p_i : S_1 \times \ldots \times S_n \to \mathbb{R}$,

for each player i.

We study strategic games under the following basic assumptions:

- players choose their strategies *simultaneously*; subsequently each player receives a payoff from the resulting joint strategy,
- each player is ***rational***, which means that his objective is to maximize his payoff,
- players have ***common knowledge*** of the game and of each others' rationality.[1]

Here are three classic examples of strategic two-player games to which we shall return in a moment. We represent such games in the form of a bimatrix, the entries of which are the corresponding payoffs to the row and column players. So for instance in the Prisoner's Dilemma game, when the row player chooses C (cooperate) and the column player chooses D (defect),

[1] Intuitively, common knowledge of some fact means that everybody knows it, everybody knows that everybody knows it, etc. This notion can be formalized using epistemic logic.

then the payoff for the row player is 0 and the payoff for the column player is 3.

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Battle of the Sexes

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies

	H	T
H	$1,-1$	$-1,\ \ 1$
T	$-1,\ \ 1$	$1,-1$

We introduce now some basic notions that will allow us to discuss and analyze strategic games in a meaningful way. Fix a strategic game

$$(S_1, \ldots, S_n, p_1, \ldots, p_n).$$

We denote $S_1 \times \ldots \times S_n$ by S, call each element $s \in S$ a ***joint strategy***, or a ***strategy profile***, denote the ith element of s by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. Finally, we abbreviate $\times_{j \neq i} S_j$ to S_{-i} and use the '$_{-i}$' notation for other sequences and Cartesian products.

We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if

$$\forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Next, we call a joint strategy s a ***Nash equilibrium*** if each s_i is a best response to s_{-i}, that is, if

$$\forall i \in \{1, \ldots, n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

So a joint strategy is a Nash equilibrium if no player can achieve a higher payoff by *unilaterally* switching to another strategy. Intuitively, a Nash equilibrium is a situation in which each player is a posteriori satisfied with his choice.

Let us return now the three above introduced games.

Re: Prisoner's Dilemma

The Prisoner's Dilemma game has a unique Nash equilibrium, namely (D, D). One of the peculiarities of this game is that in its unique Nash equilibrium each player is worse off than in the outcome (C, C). We shall return to this game once we have more tools to study its characteristics.

To clarify the importance of this game we now provide a couple of simple interpretations of it. The first one, due to Aumann, is the following.

> Each player decides whether he will receive 1000 dollars or the other will receive 2000 dollars. The decisions are simultaneous and independent.

So the entries in the bimatrix of the Prisoner's Dilemma game refer to the thousands of dollars each player will receive. For example, if the row player asks to give 2000 dollars to the other player, and the column player asks for 1000 dollar for himself, the row player gets nothing while column player gets 3000 dollars. This contingency corresponds to the 0,3 entry in the bimatrix.

The original interpretation of this game that explains its name refers to the following story.

> Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done (C), or not to confess (N).
>
> If they both do not confess, then the district attorney states he will book them on some very minor trumped-up charge such as petty larceny or illegal possession of weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor

will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped at him.

This is represented by the following bimatrix, in which each negative entry, for example -1, corresponds to the 1 year prison sentence ('the lenient treatment' referred to above):

	C	N
C	$-5, -5$	$-1, -8$
N	$-8, -1$	$-2, -2$

The negative numbers are used here to be compatible with the idea that each player is interested in maximizing his payoff, so, in this case, of receiving a lighter sentence. So for example, if the row suspect decides to confess, while the column suspect decides not to confess, the row suspect will get 1 year prison sentence (the 'lenient treatment'), the other one will get 8 years of prison (' "the book" slapped at him').

Many other natural situations can be viewed as a Prisoner's Dilemma game. This allows us to explain the underlying, undesidered phenomena.

Consider for example the arms race. For each of two warring, equally strong countries, it is beneficial not to arm instead of to arm. Yet both countries end up arming themselves. As another example consider a couple seeking a divorce. Each partner can choose an inexpensive (bad) or an expensive (good) layer. In the end both partners end up choosing expensive lawyers. Next, suppose that two companies produce a similar product and may choose between low and high advertisement costs. Both end up heavily advertising.

Re: Matching Pennies game

Next, consider the Matching Pennies game. This game formalizes a game that used to be played by children. Each of two children has a coin and simultaneously shows heads (H) or tails (T). If the coins match then the first child wins, otherwise the second child wins. This game has no Nash equilibrium. This corresponds to the intuition that for no outcome both players are satisfied. Indeed, in each outcome the losing player regrets his choice. Moreover, the sum of the payoffs is always 0. Such games, unsurprisingly, are called ***zero sum games*** and we shall return to them later. Also, we shall return to this game once we have introduced mixed strategies.

Re: Battle of the Sexes game

Finally, consider the Battle of the Sexes game. The interpretation of this game is as follows. A couple has to decide whether to go out for a football match (F) or a ballet (B). The man, the row player prefers a football match over the ballet, while the woman, the column player, the other way round. Moreover, each of them prefers to go out together than to end up going out separately. This game has two Nash equilibria, namely (F, F) and (B, B). Clearly, there is a problem how the couple should choose between these two satisfactory outcomes. Games of this type are called ***coordination games***.

Obviously, all three games are very simplistic. They deal with two players and each player has to his disposal just two strategies. In what follows we shall introduce many interesting examples of strategic games. Some of them will deal with many players and some games will have several, sometimes an infinite number of strategies.

To close this chapter we consider two examples of more interesting games, one for two players and another one for an arbitrary number of players.

Example 1 (Traveler's dilemma)

Suppose that two travellers have identical luggage, for which they both paid the same price. Their luggage is damaged (in an identical way) by an airline. The airline offers to recompense them for their luggage. They may ask for any dollar amount between \$2 and \$100. There is only one catch. If they ask for the same amount, then that is what they will both receive. However, if they ask for different amounts —say one asks for $\$m$ and the other for $\$m'$, with $m < m'$— then whoever asks for $\$m$ (the lower amount) will get $\$(m + 2)$, while the other traveller will get $\$(m - 2)$. The question is: what amount of money should each traveller ask for?

We can formalize this problem as a two-player strategic game, with the set $\{2, \ldots, 100\}$ of natural numbers as possible strategies. The following payoff function[2] formalizes the conditions of the problem:

$$p_i(s) := \begin{cases} s_i & \text{if } s_i = s_{-i} \\ s_i + 2 & \text{if } s_i < s_{-i} \\ s_{-i} - 2 & \text{otherwise} \end{cases}$$

It is easy to check that $(2, 2)$ is a Nash equilibrium. To check for other Nash equilibria consider any other combination of strategies (s_i, s_{-i}) and

[2]We denote in two-player games the opponent of player i by $-i$, instead of $3 - i$.

suppose that player i submitted a larger or equal amount, i.e., $s_i \geq s_{-i}$. Then player's i payoff is s_{-i} if $s_i = s_{-i}$ or $s_{-i} - 2$ if $s_i > s_{-i}$.

In the first case he will get a strictly higher payoff, namely $s_{-i} + 1$, if he submits instead the amount $s_{-i} - 1$. (Note that $s_i = s_{-i}$ and $(s_i, s_{-i}) \neq (2, 2)$ implies that $s_{-i} - 1 \in \{2, \ldots, 100\}$.) In turn, in the second case he will get a strictly higher payoff, namely s_{-i}, if he submits instead the amount s_{-i}.

So in each joint strategy $(s_i, s_{-i}) \neq (2, 2)$ at least one player has a strictly better alternative, i.e., his strategy is not a best response. This means that $(2, 2)$ is a unique Nash equilibrium. This is a paradoxical conclusion, if we recall that informally a Nash equilibrium is a state in which both players are satisfied with their choice. □

Example 2 Consider the following ***beauty contest game***. In this game there are $n > 2$ players, each with the set of strategies equal $\{1, \ldots, 100\}$, Each player submits a number and the payoff to each player is obtained by splitting 1 equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average. For example, if the submissions are $29, 32, 29$, then the payoffs are respectively $\frac{1}{2}, 0, \frac{1}{2}$.

Finding Nash equilibria of this game is not completely straightforward. At this stage we only observe that the joint strategy $(1, \ldots, 1)$ is clearly a Nash equilibrium. We shall answer the question of whether there are more Nash equilibria once we introduce some tools to analyze strategic games. □

Exercise 1 Find all Nash equilibria in the following games:

Stag hunt

	S	R
S	$2, 2$	$0, 1$
R	$1, 0$	$1, 1$

Coordination

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$1, 1$

Pareto Coordination

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Hawk-dove

	H	D
H	$0, 0$	$3, 1$
D	$1, 3$	$2, 2$

□

Exercise 2 Watch the following video `https://www.youtube.com/watch?v=p3Uos2fzIJ0`. Define the underlying game. What are its Nash equilibria?

Exercise 3 Consider the following ***inspection game***.

There are two players: a worker and the boss. The worker can either Shirk or put an Effort, while the boss can either Inspect or Not. Finding a shirker has a benefit b while the inspection costs c, where $b > c > 0$. So if the boss carries out an inspection his benefit is $b - c > 0$ if the worker shirks and $-c < 0$ otherwise.

The worker receives 0 if he shirks and is inspected, and g if he shirks and is not found. Finally, the worker receives w, where $g > w > 0$ if he puts in the effort.

This leads to the following bimatrix:

	I	N
S	$0, b-c$	$g, 0$
E	$w, -c$	$w, 0$

Analyze the best responses in this game. What can we conclude from it about the Nash equilibria of this game?

□

Chapter 2

Social Optima

To discuss strategic games in a meaningful way we need to introduce further, natural, concepts. Fix a strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

We call a joint strategy s a ***Pareto efficient outcome*** if for no joint strategy s'

$$\forall i \in \{1, \ldots, n\}\ p_i(s') \geq p_i(s) \text{ and } \exists i \in \{1, \ldots, n\}\ p_i(s') > p_i(s).$$

That is, a joint strategy is a Pareto efficient outcome if no joint strategy is both a weakly better outcome for all players and a strictly better outcome for some player.

Further, given a joint strategy s we call the sum $\sum_{j=1}^{n} p_j(s)$ the ***social welfare*** of s. Next, we call a joint strategy s a ***social optimum*** if its social welfare is maximal.

Clearly, if s is a social optimum, then s is Pareto efficient. The converse obviously does not hold. Indeed, in the Prisoner's Dilemma game the joint strategis (C, D) and (D, C) are both Pareto efficient, but their social welfare is not maximal. Note that (D, D) is the only outcome that is not Pareto efficient. The social optimum is reached in the strategy profile (C, C). In contrast, the social welfare is smallest in the Nash equilibrium (D, D).

This discrepancy between Nash equilibria and Pareto efficient outcomes is absent in the Battle of Sexes game. Indeed, here both concepts coincide.

The tension between Nash equilibria and Pareto efficient outcomes present in the Prisoner's Dilemma game occurs in several other natural games. It forms one of the fundamental topics in the theory of strategic games. In this chapter we shall illustrate this phenomenon by a number of examples.

Example 3 (Prisoner's Dilemma for n players)
First, the Prisoner's Dilemma game can be easily generalized to n players as follows. It is convenient to assume that each player has two strategies, 1, representing cooperation, (formerly C) and 0, representing defection, (formerly D). Then, given a joint strategy s_{-i} of the opponents of player i, $\sum_{j \neq i} s_j$ denotes the number of 1 strategies in s_{-i}. Denote by $\mathbf{1}$ the joint strategy in which each strategy equals 1 and similarly with $\mathbf{0}$.

We put

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

Note that for $n = 2$ we get the original Prisoner's Dilemma game.

It is easy to check that the strategy profile $\mathbf{0}$ is the unique Nash equilibrium in this game. Indeed, in each other strategy profile a player who chose 1 (cooperate) gets a higher payoff when he switches to 0 (defect).

Finally, note that the social welfare in $\mathbf{1}$ is $2n(n-1)$, which is strictly more than n, the social welfare in $\mathbf{0}$. We now show that $2n(n-1)$ is the social optimum. To this end it suffices to note that if a single player switches from 0 to 1, then his payoff decreases by 1 but the payoff of each other player increases by 2, and hence the social welfare increases. $\square$

The next example deals with the depletion of ***common resources***, which in economics are goods that are not *excludable* (people cannot be prevented from using them) but are *rival* (one person's use of them diminishes another person's enjoyment of it). Examples are congested toll-free roads, fish in the ocean, or the environment. The overuse of such common resources leads to their destruction. This phenomenon is called the ***tragedy of the commons***.

One way to model it is as a Prisoner's dilemma game for n players. But such a modeling is too crude as it does not reflect the essential characteristics of the problem. We provide two more adequate modeling of it, one for the case of a binary decision (for instance, whether to use a congested road or not), and another one for the case when one decides about the intensity of using the resource (for instance on what fraction of a lake should one fish).

Example 4 (Tragedy of the commons I)
Assume $n > 1$ players, each having to its disposal two strategies, 1 and 0 reflecting, respectively, that the player decides to use the common resource or not. If he does not use the resource, he gets a fixed payoff. Further, the users

of the resource get the same payoff. Finally, the more users of the common resource the smaller payoff for each of them gets, and when the number of users exceeds a certain threshold it is better for the other players not to use the resource.

The following payoff function realizes these assumptions:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$

Indeed, the function $F(m)/m$ is strictly decreasing. Moreover, $F(9)/9 = 0.2$, $F(10)/10 = 0.1$ and $F(11)/11 = 0$. So when there are already ten or more users of the resource it is indeed better for other players not to use the resource.

To find a Nash equilibrium of this game, note that given a strategy profile s with $m = \sum_{j=1}^{n} s_j$ player i profits from switching from s_i to another strategy in precisely two circumstances:

- $s_i = 0$ and $F(m+1)/(m+1) > 0.1$,
- $s_i = 1$ and $F(m)/m < 0.1$.

In the first case we have $m + 1 < 10$ and in the second case $m > 10$.

Hence when $n < 10$ the only Nash equilibrium is when all players use the common resource and when $n \geq 10$ then s is a Nash equilibrium when either 9 or 10 players use the common resource.

Assume now that $n \geq 10$. Then in a Nash equilibrium s the players who use the resource receive the payoff 0.2 (when $m = 9$) or 0.1 (when $m = 10$). So the maximum social welfare that can be achieved in a Nash equilibrium is $0.1(n - 9) + 1.8 = 0.1n + 0.9$.

To find a strategy profile in which social optimum is reached with the largest social welfare we need to find m for which the function $0.1(n - m) + F(m)$ reaches the maximum. Now, $0.1(n - m) + F(m) = 0.1n + m - 0.1m^2$ and by elementary calculus we find that $m = 5$ for which $0.1(n-m)+F(m) = 0.1n+2.5$. So the social optimum is achieved when 5 players use the common resource. □

Example 5 (Tragedy of the commons II)
Assume $n > 1$ players, each having to its disposal an infinite set of strategies that consists of the real interval $[0, 1]$. View player's strategy as its chosen fraction of the common resource. Then the following payoff function reflects the fact that player's enjoyment of the common resource depends positively from his chosen fraction of the resource and negatively from the total fraction of the common resource used by all players:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The second alternative reflects the phenomenon that if the total fraction of the common resource by all players exceeds a feasible level, here 1, then player's enjoyment of the resource becomes zero. We can write the payoff function in a more compact way as

$$p_i(s) := \max(0, s_i(1 - \sum_{j=1}^{n} s_j)).$$

To find a Nash equilibrium of this game, fix $i \in \{1, \ldots, n\}$ and s_{-i} and denote $\sum_{j \neq i} s_j$ by t. Then $p_i(s_i, s_{-i}) = \max(0, s_i(1 - t - s_i))$.

By elementary calculus player's i payoff becomes maximal when $s_i = \frac{1-t}{2}$. This implies that when for all $i \in \{1, \ldots, n\}$ we have

$$s_i = \frac{1 - \sum_{j \neq i} s_j}{2},$$

then s is a Nash equilibrium. This system of n linear equations has a unique solution $s_i = \frac{1}{n+1}$ for $i \in \{1, \ldots, n\}$. In this strategy profile each player's payoff is $\frac{1-n/(n+1)}{n+1} = \frac{1}{(n+1)^2}$, so its social welfare is $\frac{n}{(n+1)^2}$.

There are other Nash equilibria. Indeed, suppose that for all $i \in \{1, \ldots, n\}$ we have $\sum_{j \neq i} s_j \geq 1$, which is the case for instance when $s_i = \frac{1}{n-1}$ for $i \in \{1, \ldots, n\}$. It is straightforward to check that each such strategy profile is a Nash equilibrium in which each player's payoff is 0 and hence the social welfare is also 0. It is easy to check that no other Nash equilibria exist.

To find a strategy profile in which social optimum is reached fix a strategy profile s and let $t := \sum_{j=1}^{n} s_j$. First note that if $t > 1$, then the social welfare is 0. So assume that $t \leq 1$. Then $\sum_{j=1}^{n} p_j(s_j) = t(1 - t)$. By elementary calculus this expression becomes maximal precisely when $t = \frac{1}{2}$ and then it equals $\frac{1}{4}$.

Now, for all $n > 1$ we have $\frac{n}{(n+1)^2} < \frac{1}{4}$. So the social welfare of each solution for which $\sum_{j=1}^{n} s_j = \frac{1}{2}$ is superior to the social welfare of the Nash equilibria. In particular, no such strategy profile is a Nash equilibrium.

In conclusion, the social welfare is maximal, and equals $\frac{1}{4}$, when precisely half of the common resource is used. In contrast, in the 'best' Nash equilibrium the social welfare is $\frac{n}{(n+1)^2}$ and the fraction $\frac{n}{n+1}$ of the common resource is used. So when the number of players increases, the social welfare of the best Nash equilibrium becomes arbitrarily small, while the fraction of the common resource being used becomes arbitrarily large. $\square$

The analysis carried out in the above two examples reveals that for the adopted payoff functions the common resource will be overused, to the detriment of the players (society). The same conclusion can be drawn for a much larger of class payoff functions that properly reflect the characteristics of using a common resource.

Example 6 (Cournot competition)
This example deals with a situation in which n companies independently decide their production levels of a given product. The price of the product is a linear function that depends negatively on the total output.

We model it by means of the following strategic game. We assume that for each player i:

- his strategy set is $\mathbb{R}_+$,
- his payoff function is defined by

$$p_i(s) := s_i(a - b\sum_{j=1}^{n} s_j) - cs_i$$

for some given a, b, c, where $a > c$ and $b > 0$.

Let us explain this payoff function. The price of the product is represented by the expression $a - b\sum_{j=1}^{n} s_j$, which, thanks to the assumption $b > 0$, indeed depends negatively on the total output, $\sum_{j=1}^{n} s_j$. Further, cs_i is the production cost corresponding to the production level s_i. So we assume for simplicity that the production cost functions are the same for all companies.

Further, note that if $a \leq c$, then the payoffs would be always negative or zero, since $p_i(s) = (a-c)s_i - bs_i \sum_{j=1}^{n} s_j$. This explains the assumption that $a > c$. For simplicity we do allow a possibility that the prices are negative, but see Exercise 5. The assumption $c > 0$ is obviously meaningful but not needed.

To find a Nash equilibrium of this game fix $i \in \{1, \ldots, n\}$ and s_{-i} and denote $\sum_{j \neq i} s_j$ by t. Then $p_i(s_i, s_{-i}) = s_i(a - c - bt - bs_i)$. By elementary calculus player's i payoff becomes maximal iff

$$s_i = \frac{a-c}{2b} - \frac{t}{2}.$$

This implies that s is a Nash equilibrium iff for all $i \in \{1, \ldots, n\}$

$$s_i = \frac{a-c}{2b} - \frac{\sum_{j \neq i} s_j}{2}.$$

One can check that this system of n linear equations has a unique solution, $s_i = \frac{a-c}{(n+1)b}$ for $i \in \{1, \ldots, n\}$. So this is a unique Nash equilibrium of this game.

Note that for these values of s_is the price of the product is

$$a - b \sum_{j=1}^{n} s_j = a - b\frac{n(a-c)}{(n+1)b} = \frac{a+nc}{n+1}.$$

To find the social optimum let $t := \sum_{j=1}^{n} s_j$. Then $\sum_{j=1}^{n} p_j(s) = t(a - c - bt)$. By elementary calculus this expression becomes maximal precisely when $t = \frac{a-c}{2b}$. So s is a social optimum iff $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$. The price of the product in a social optimum is $a - b\frac{a-c}{2b} = \frac{a+c}{2}$.

Now, the assumption $a > c$ implies that $\frac{a+c}{2} > \frac{a+nc}{n+1}$. So we see that the price in the social optimum is strictly higher than in the Nash equilibrium. This can be interpreted as a statement that the competition between the producers of the product drives its price down, or alternatively, that the cartel between the producers leads to higher profits for them (i.e., higher social welfare), at the cost of a higher price. So in this example reaching the social optimum is not a desirable state of affairs. The reason is that in our analysis we focussed only on the profits of the producers and omitted the customers.

Further notice that when n, so the number of companies, increases, the price $\frac{a+nc}{n+1}$ in the Nash equilibrium decreases. This corresponds to the intuition that increased competition is beneficial for the customers. Note also that in the limit the price in the Nash equilibrium converges to the production cost c.

Finally, let us compare the social welfare in the unique Nash equilibrium and a social optimum. We just noted that for $t := \sum_{j=1}^{n} s_j$ we have $\sum_{j=1}^{n} p_j(s) = t(a-c-bt)$, and that for the unique Nash equilibrium s we have $s_i = \frac{a-c}{(n+1)b}$ for $i \in \{1,\ldots,n\}$. So $t = \frac{a-c}{b}\frac{n}{n+1}$ and consequently

$$\begin{aligned}\sum_{j=1}^{n} p_j(s) &= \frac{a-c}{b}\frac{n}{n+1}(a-c-(a-c)\frac{n}{n+1}) \\ = \quad & \frac{a-c}{b}\frac{n}{n+1}\frac{1}{n+1}(a-c) = \frac{(a-c)^2}{b}\frac{n}{(n+1)^2}\end{aligned}$$

This shows that the social welfare in the unique Nash equilibrium converges to 0 when n, the number of companies, goes to infinity. This can be interpreted as a statement that the increased competition between producers results in their profits becoming arbitrary small.

In contrast, the social welfare in each social optimum remains constant. Indeed, we noted that s is a social welfare iff $t = \frac{a-c}{2b}$ where $t := \sum_{j=1}^{n} s_j$. So for each social welfare s we have

$$\sum_{j=1}^{n} p_j(s) = t(a-c-bt) = \frac{a-c}{2b}(a-c-\frac{a-c}{2}) = \frac{(a-c)^2}{4b}.$$

$\square$

While the last two examples refer to completely different scenarios, their mathematical analysis is very similar. Their common characteristics is that in both examples the payoff functions can be written as $f(s_i, \sum_{j=1}^{n} s_j)$, where f is increasing in the first argument and decreasing in the second argument.

Exercise 4 Prove that in the game discussed in Example 5 indeed no other Nash equilibria exist apart of the mentioned ones. $\square$

Exercise 5 Modify the game from Example 6 by considering the following payoff functions:

$$p_i(s) := s_i \max(0, a - b\sum_{j=1}^{n} s_j) - cs_i.$$

Compute the Nash equilibria of this game.
Hint. Proceed as in Example 5. $\square$