Chapter 3

Strict Dominance

Let us return now to our analysis of an arbitrary strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Let s_i, s'_i be strategies of player i. We say that s_i ***strictly dominates*** s'_i (or equivalently, that s'_i is ***strictly dominated by*** s_i) if

$$\forall s_{-i} \in S_{-i} \; p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}).$$

Further, we say that s_i is ***strictly dominant*** if it strictly dominates all other strategies of player i.

Clearly, a rational player will not choose a strictly dominated strategy. As an illustration let us return to the Prisoner's Dilemma. In this game for each player C (cooperate) is a strictly dominated strategy. So the assumption of players' rationality implies that each player will choose strategy D (defect). That is, we can predict that rational players will end up choosing the joint strategy (D, D) in spite of the fact that the Pareto efficient outcome (C, C) yields for each of them a strictly higher payoff.

The same holds in the Prisoner's Dilemma game for n players, where for all players i strategy 1 is strictly dominated by strategy 0, since for all $s_{-i} \in S_{-i}$ we have $p_i(0, s_{-i}) - p_i(1, s_{-i}) = 1$.

We assumed that each player is rational. So when searching for an outcome that is optimal for all players we can safely remove strategies that are strictly dominated by some other strategy. This can be done in a number of ways. For example, we could remove all or some strictly dominated strategies simultaneously, or start removing them in a round Robin fashion starting with, say, player 1. To discuss this matter more rigorously we introduce the notion of a restriction of a game.

Given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and (possibly empty) sets of strategies $R_1, \ldots, R_n$ such that $R_i \subseteq S_i$ for $i \in \{1, \ldots, n\}$ we say that $R := (R_1, \ldots, R_n, p_1, \ldots, p_n)$ is a ***restriction*** of G. Here of course we view each p_i as a function on the subset $R_1 \times \ldots \times R_n$ of $S_1 \times \ldots \times S_n$.

In what follows, given a restriction R we denote by R_i the set of strategies of player i in R. Further, given two restrictions R and R' of G we write $R' \subseteq R$ when $\forall i \in \{1, \ldots, n\}\ R'_i \subseteq R_i$. We now introduce the following notion of reduction between the restrictions R and R' of G:

$$R \rightarrow_S R'$$

when $R \neq R'$, $R' \subseteq R$ and

$$\forall i \in \{1, \ldots, n\}\ \forall s_i \in R_i \setminus R'_i\ \exists s'_i \in R_i\ s_i \text{ is strictly dominated in } R \text{ by } s'_i.$$

That is, $R \rightarrow_S R'$ when R' results from R by removing from it some strictly dominated strategies.

We now clarify whether a one-time elimination of (some) strictly dominated strategies can affect Nash equilibria.

Lemma 1 (Strict Elimination) *Given a strategic game G consider two restrictions R and R' of G such that $R \rightarrow_S R'$. Then*

(i) if s is a Nash equilibrium of R, then it is a Nash equilibrium of R',

(ii) if G is finite and s is a Nash equilibrium of R', then it is a Nash equilibrium of R.

At the end of this chapter we shall clarify why in (ii) the restriction to finite games is necessary.

Proof.
(i) For each player the set of his strategies in R' is a subset of the set of his strategies in R. So to prove that s is a Nash equilibrium of R' it suffices to prove that no strategy constituting s is eliminated. Suppose otherwise. Then some s_i is eliminated, so for some $s'_i \in R_i$

$$p_i(s'_i, s''_{-i}) > p_i(s_i, s''_{-i}) \text{ for all } s''_{-i} \in R_{-i}.$$

In particular

$$p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}),$$

so s is not a Nash equilibrium of R.

(ii) Suppose s is not a Nash equilibrium of R. Then for some $i \in \{1, \ldots, n\}$ strategy s_i is not a best response of player i to s_{-i} in R.

Let $s'_i \in R_i$ be a best response of player i to s_{-i} in R (which exists since R_i is finite). The strategy s'_i is eliminated since s is a Nash equilibrium of R'. So for some $s^*_i \in R_i$

$$p_i(s^*_i, s''_{-i}) > p_i(s'_i, s''_{-i}) \text{ for all } s''_{-i} \in R_{-i}.$$

In particular

$$p_i(s^*_i, s_{-i}) > p_i(s'_i, s_{-i}),$$

which contradicts the choice of s'_i. □

In general an elimination of strictly dominated strategies is not a one step process; it is an iterative procedure. Its use is justified by the assumption of common knowledge of rationality.

Example 7 Consider the following game:

	L	M	R
T	$3,0$	$2,1$	$1,0$
C	$2,1$	$1,1$	$1,0$
B	$0,1$	$0,1$	$0,0$

Note that B is strictly dominated by T and R is strictly dominated by M. By eliminating these two strategies we get:

	L	M
T	$3,0$	$2,1$
C	$2,1$	$1,1$

Now C is strictly dominated by T, so we get:

	L	M
T	$3,0$	$2,1$

In this game L is strictly dominated by M, so we finally get:

	M
T	$2,1$

□

This brings us to the following notion, where given a binary relation $\rightarrow$ we denote by $\rightarrow^*$ its transitive reflexive closure. Consider a strategic game G. Suppose that $G \rightarrow_S^* R$, i.e., R is obtained by an iterated elimination of strictly dominated strategies, in short ***IESDS***, starting with G.

- If for no restriction R' of G, $R \rightarrow_S R'$ holds, we say that R is ***an outcome of IESDS from*** G.
- If each player is left in R with exactly one strategy, we say that G ***is solved by IESDS***.

The following result then clarifies the relation between the IESDS and Nash equilibrium.

Theorem 2 (IESDS) *Suppose that G' is an outcome of IESDS from a strategic game G.*

(i) If s is a Nash equilibrium of G, then it is a Nash equilibrium of G'.

(ii) If G is finite and s is a Nash equilibrium of G', then it is a Nash equilibrium of G.

(iii) If G is finite and solved by IESDS, then the resulting joint strategy is a unique Nash equilibrium.

Proof. By the Strict Elimination Lemma 1. □

We also have the following observation.

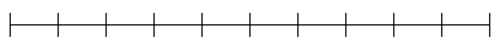
Note 3 (Strict Dominance) *Consider a strategic game G.*

Suppose that s is a joint strategy such that each s_i is a strictly dominant strategy. Then it is a unique Nash equilibrium of G.

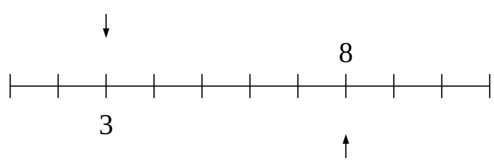
Proof. By assumption s is a Nash equilibrium. Take now some $s' \neq s$. For some i we have $s'_i \neq s_i$. By assumption $p_i(s_i, s'_{-i}) > p_i(s'_i, s'_{-i})$, where p_i is the payoff function of player i. So s' is not a Nash equilibrium. □

Example 8 A nice example of a game that is solved by IESDS is the ***location game***. Assume that that the players are two vendors who simultaneously choose a location. Then the customers choose the closest vendor. The profit for each vendor equals the number of customers it attracted.

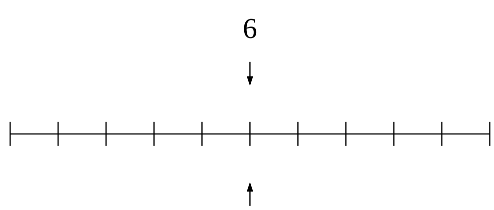
To be more specific we assume that the vendors choose a location from the set $\{1, \ldots, n\}$ of natural numbers, viewed as points on a real line, and that at each location there is exactly one customer. For example, for $n = 11$ we have 11 locations:



and when the players choose respectively the locations 3 and 8:



we have $p_1(3, 8) = 5$ and $p_2(3, 8) = 6$. When the vendors 'share' a customer, for instance when they both choose the location 6:



they end up with a fractional payoff, in this case $p_1(6, 6) = 5.5$ and $p_1(6, 6) = 5.5$.

In general, we have the following game:

- each set of strategies consists of the set $\{1, \ldots, n\}$,
- each payoff function p_i is defined by:

$$p_i(s_i, s_{-i}) := \begin{cases} \dfrac{s_i + s_{-i} - 1}{2} & \text{if } s_i < s_{-i} \\ n - \dfrac{s_i + s_{-i} - 1}{2} & \text{if } s_i > s_{-i} \\ \dfrac{n}{2} & \text{if } s_i = s_{-i} \end{cases}$$

It is easy to check that for $n = 2k + 1$ this game is solved by k rounds of IESDS, and that each player is left with the 'middle' strategy k. In each round both 'outer' strategies are eliminated, so first 1 and n, then 2 and $n - 1$, and so on. □

There is one more natural question that we left so far unanswered. Is the outcome of an iterated elimination of strictly dominated strategies unique, or in the game theory parlance: is strict dominance ***order independent***? The answer is positive.

Theorem 4 (Order Independence I) *Given a finite strategic game all iterated eliminations of strictly dominated strategies yield the same outcome.*

Proof. See the Appendix of this Chapter. □

The above result does not hold for infinite strategic games.

Example 9 Consider a game in which the set of strategies for each player is the set of natural numbers. The payoff to each player is the number (strategy) he selected.

Note that in this game every strategy is strictly dominated. Consider now three ways of using IESDS:

- by removing in one step all strategies that are strictly dominated,
- by removing in one step all strategies different from 0 that are strictly dominated,
- by removing in each step exactly one strategy, for instance the least even strategy.

In the first case we obtain the restriction with the empty strategy sets, in the second one we end up with the restriction in which each player has just one strategy, 0, and in the third case we obtain an infinite sequence of reductions. □

The above example also shows that in the limit of an infinite sequence of reductions different outcomes can be reached. So for infinite games the definition of the order independence has to be modified.

The above example also shows that in the Strict Elimination 1(*ii*) and the IESDS Theorem 2(*ii*) and (*iii*) we cannot drop the assumption that the game is finite. Indeed, the above infinite game has no Nash equilibria, while the game in which each player has exactly one strategy has a Nash equilibrium.

Exercise 6

(i) What is the outcome of IESDS in the location game with an even number of locations?

(ii) Modify the location game from Example 8 to a game for three players. Prove that this game has no Nash equilibrium.

(iii) Define a modification of the above game for three players to the case when the set of possible locations (both for the vendors and the customers) forms a circle. Find the set of Nash equilibria. □

Appendix

We provide here the proof of the Order Independence I Theorem 4. Conceptually it is useful to carry out these consideration in a more general setting. We assume an initial strategic game

$$G := (G_1, \ldots, G_n, p_1, \ldots, p_n).$$

By a ***dominance relation*** D we mean a function that assigns to each restriction R of G a subset D_R of $\bigcup_{i=1}^{n} R_i$. Instead of writing $s_i \in D_R$ we say that s_i ***is*** D***-dominated in*** R.

Given two restrictions R and R' we write $R \rightarrow_D R'$ when $R \neq R', R' \subseteq R$ and

$$\forall i \in \{1, \ldots, n\} \; \forall s_i \in R_i \setminus R'_i \;\; s_i \text{ is } D\text{-dominated in } R.$$

Clearly being strictly dominated by another strategy is an example of a dominance relation and $\rightarrow_S$ is an instance of $\rightarrow_D$.

An ***outcome*** of an iteration of $\rightarrow_D$ starting in a game G is a restriction R that can be reached from G using $\rightarrow_D$ in finitely many steps and such that for no R', $R \rightarrow_D R'$ holds.

We call a dominance relation D

- ***order independent*** if for all initial finite games G all iterations of $\rightarrow_D$ starting in G yield the same final outcome,

- ***hereditary*** if for all initial games G, all restrictions R and R' such that $R \rightarrow_D R'$ and a strategy s_i in R'

  $$s_i \text{ is } D\text{-dominated in } R \text{ implies that } s_i \text{ is } D\text{-dominated in } R'.$$

We now establish the following general result.

Theorem 5 *Every hereditary dominance relation is order independent.*

To prove it we introduce the notion of an ***abstract reduction system***. It is simply a pair $(A, \rightarrow)$ where A is a set and $\rightarrow$ is a binary relation on A. Recall that $\rightarrow^*$ denotes the transitive reflexive closure of $\rightarrow$.

- We say that b is a $\rightarrow$***-normal form of*** a if $a \rightarrow^* b$ and no c exists such that $b \rightarrow c$, and omit the reference to $\rightarrow$ if it is clear from the context. If every element of A has a unique normal form, we say that $(A, \rightarrow)$ (or just $\rightarrow$ if A is clear from the context) satisfies the ***unique normal form property***.

- We say that $\rightarrow$ is ***weakly confluent*** if for all $a, b, c \in A$

  $$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \end{array}$$

  implies that for some $d \in A$

  $$\begin{array}{ccc} b & & c \\ \searrow^{*} & & {}^{*}\swarrow \\ & d & \end{array}$$

We need the following crucial lemma.

Lemma 6 (Newman) *Consider an abstract reduction system* $(A, \rightarrow)$ *such that*

- *no infinite $\rightarrow$ sequences exist,*
- *$\rightarrow$ is weakly confluent.*

Then $\rightarrow$ satisfies the unique normal form property.

Proof. By the first assumption every element of A has a normal form. To prove uniqueness call an element a *ambiguous* if it has at least two different normal forms. We show that for every ambiguous a some ambiguous b exists such that $a \rightarrow b$. This proves absence of ambiguous elements by the first assumption.

So suppose that some element a has two distinct normal forms n_1 and n_2. Then for some b, c we have $a \rightarrow b \rightarrow^* n_1$ and $a \rightarrow c \rightarrow^* n_2$. By weak confluence some d exists such that $b \rightarrow^* d$ and $c \rightarrow^* d$. Let n_3 be a normal form of d. It is also a normal form of b and of c. Moreover $n_3 \neq n_1$ or $n_3 \neq n_2$. If $n_3 \neq n_1$, then b is ambiguous and $a \rightarrow b$. And if $n_3 \neq n_2$, then c is ambiguous and $a \rightarrow c$. □

Proof of Theorem 5.

Take a hereditary dominance relation D. Consider a restriction R. Suppose that $R \rightarrow_D R'$ for some restriction R'. Let R'' be the restriction of R obtained by removing all strategies that are D-dominated in R.

We have $R'' \subseteq R'$. Assume that $R' \neq R''$. Choose an arbitrary strategy s_i such that $s_i \in R'_i \setminus R''_i$. So s_i is D-dominated in R. By the hereditarity of D, s_i is also D-dominated in R'. This shows that $R' \rightarrow_D R''$.

So we proved that either $R' = R''$ or $R' \rightarrow_D R''$, i.e., that $R' \rightarrow_D^* R''$. This implies that $\rightarrow_D$ is weakly confluent. It suffices now to apply Newman's Lemma 6. □

To apply this result to strict dominance we establish the following fact.

Lemma 7 (Hereditarity I) *The relation of being strictly dominated is hereditary on the set of restrictions of a given finite game.*

Proof. Suppose a strategy $s_i \in R'_i$ is strictly dominated in R and $R \rightarrow_S R'$. The initial game is finite, so there exists in R_i a strategy s'_i that strictly dominates s_i in R and is not strictly dominated in R. Then s'_i is not eliminated in the step $R \rightarrow_S R'$ and hence is a strategy in R'_i. But $R' \subseteq R$, so s'_i also strictly dominates s_i in R'. □

The promised proof is now immediate.

Proof of the Order Independence I Theorem 4.

By Theorem 5 and the Hereditarity I Lemma 7. □