Solution to Assignment 5

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Consider the network given in Figure 1. The delays on the road segments are either constant (4 or 5) or equal to the number of drivers who chose the segment (denoted by T).

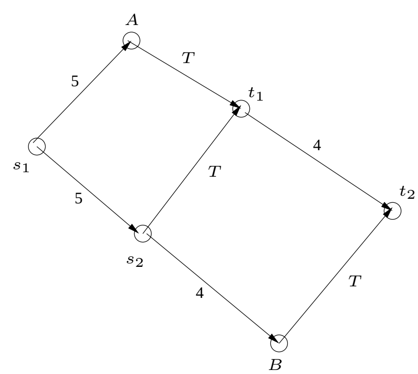

Figure 1: A network

There are 6 drivers who need to choose a road from s_1 to t_1 and 6 drivers who need to choose a road from s_2 to t_2. So each of the drivers in the first set has two strategies, corresponding respectively to the roads $s_1 \rightarrow A \rightarrow t_1$ and $s_1 \rightarrow s_2 \rightarrow t_1$, while each of the drivers in the second set has two strategies, corresponding respectively to the roads $s_2 \rightarrow t_1 \rightarrow t_2$ and $s_2 \rightarrow B \rightarrow t_2$.

We now determine the Nash equilibria and the social optima in the resulting congestion game. Consider a joint strategy. Denote by

- T_1 the number of drivers who took the road $s_1 \rightarrow A \rightarrow t_1$,
- T_2 the number of drivers who took the road $s_1 \rightarrow s_2 \rightarrow t_1$,
- T_3 the number of drivers who took the road $s_2 \rightarrow t_1 \rightarrow t_2$,
- T_4 the number of drivers who took the road $s_2 \rightarrow B \rightarrow t_2$.

By assumption we have

$$T_1 + T_2 = 6, \; T_3 + T_4 = 6.$$

Note that $T_2{+}T_3$ is then the number of drivers who took the road segment $s_2 \to t_1$. Consequently, the considered strategy is a Nash equilibrium iff the following constraints are satisfied for the drivers who need to choose a road from s_1 to t_1:

- for changing the road $s_1 \to s_2 \to t_1$ to $s_1 \to A \to t_1$:
  $T_2 > 0 \to 5 + T_1 + 1 \geq 5 + T_2 + T_3$,
- for changing the road $s_1 \to A \to t_1$ to $s_1 \to s_2 \to t_1$:
  $T_1 > 0 \to 5 + T_2 + T_3 + 1 \geq 5 + T_1$,

and the following constraints are satisfied for the drivers who need to choose a road from s_2 to t_2:

- for changing the road $s_2 \to B \to t_2$ to $s_2 \to t_1 \to t_2$:
  $T_4 > 0 \to T_2 + T_3 + 4 + 1 \geq 4 + T_4$,
- for changing the road $s_2 \to t_1 \to t_2$ to $s_2 \to B \to t_2$:
  $T_3 > 0 \to 4 + T_4 + 1 \geq T_2 + T_3 + 4$.

Further, the social cost of the considered joint strategy equals

$$(5 + T_1)T_1 + (5 + T_2 + T_3)T_2 + (T_2 + T_3 + 4)T_3 + (4 + T_4)T_4.$$

One can check (we did it using the programming language ECL^{i}PS^e) that there are three ways of satisfying the above constraints:

- $T_1 = 3, T_2 = 3, T_3 = 1, T_4 = 5$, with the social cost 104,
- $T_1 = 4, T_2 = 2, T_3 = 2, T_4 = 4$, with the social cost 102,
- $T_1 = 5, T_2 = 1, T_3 = 3, T_4 = 3$, with the social cost 104.

The second Nash equilibrium is also a social optimum. Consequently, the price of anarchy of this game equals $\frac{104}{102}$, while the price of stability equals 1.

Suppose now that one adds to the network a road $t_1 \to B$ with delay 0. The resulting network is drawn in Figure 2.

The drivers who need to choose a road from s_2 to t_2 have then three strategies. Given a joint strategy we denote now by

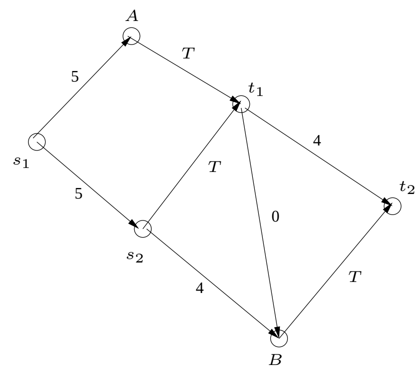

Figure 2: The new network

- T_5 the number of drivers who took the road $s_2 \to t_1 \to B \to t_2$,

and define T_1, T_2, T_3 and T_4 as before. We have then

$$T_1 + T_2 = 6,\ T_3 + T_4 + T_5 = 6.$$

Note that now the number of drivers who took the road segment $s_2 \to t_1$ equals $T_2 + T_3 + T_5$, while the number of drivers who took the road segment $B \to t_2$ now equals $T_4 + T_5$. Consequently, the considered strategy is a Nash equilibrium iff the following constraints are satisfied for the drivers who need to choose a road from s_1 to t_1:

- for changing the road $s_1 \to s_2 \to t_1$ to $s_1 \to A \to t_1$:

  $T_2 > 0 \to 5 + T_1 + 1 \geq 5 + T_2 + T_3 + T_5$,

- for changing the road $s_1 \to A \to t_1$ to $s_1 \to s_2 \to t_1$:

  $T_1 > 0 \to 5 + T_2 + T_3 + T_5 + 1 \geq 5 + T_1$.

Additionally, we have now the following six constraints for the drivers who need to choose a road from s_2 to t_2:

- for changing the road $s_2 \to B \to t_2$ to $s_2 \to t_1 \to t_2$:

  $T_4 > 0 \to T_2 + T_3 + T_5 + 4 + 1 \geq 4 + T_4 + T_5$,

- for changing the road $s_2 \to t_1 \to t_2$ to $s_2 \to B \to t_2$:

  $T_3 > 0 \to 4 + T_4 + T_5 + 1 \geq T_2 + T_3 + T_5 + 4$,

- for changing the road $s_2 \to t_1 \to B \to t_2$ to $s_2 \to t_1 \to t_2$:

  $T_5 > 0 \to T_2 + T_3 + T_5 + 4 \geq T_2 + T_3 + T_5 + 0 + T_4 + T_5$,

- for changing the road $s_2 \to t_1 \to t_2$ to $s_2 \to t_1 \to B \to t_2$:
  $T_3 > 0 \to T_2 + T_3 + T_5 + 0 + T_4 + T_5 + 1 \geq T_2 + T_3 + T_5 + 4$,
- for changing the road $s_2 \to t_1 \to B \to t_2$ to $s_2 \to B \to t_2$:
  $T_5 > 0 \to 4 + T_4 + T_5 \geq T_2 + T_3 + T_5 + 0 + T_4 + T_5$,
- for changing the road $s_2 \to B \to t_2$ to $s_2 \to t_1 \to B \to t_2$:
  $T_4 > 0 \to T_2 + T_3 + T_5 + 0 + T_4 + T_5 + 1 \geq 4 + T_4 + T_5$.

The social cost of the considered joint strategy now equals

$$(5 + T_1)T_1 + (5 + T_2 + T_3 + T_5)T_2 + (T_2 + T_3 + T_5 + 4)T_3 +$$
$$(4 + T_4 + T_5)T_4 + (T_2 + T_3 + T_5 + 0 + T_4 + T_5)T_5.$$

Further, one can check that each of previous three Nash equilibria when augmented with $T_5 = 0$ is a Nash equilibrium in the new game. However, there is now an additional Nash equilibrium, namely

- $T_1 = 5, T_2 = 1, T_3 = 2, T_4 = 3, T_5 = 1$, with the social cost 107.

One can also check that social optimum is reached in the Nash equilibrium

- $T_1 = 4, T_2 = 2, T_3 = 2, T_4 = 4, T_5 = 0$, with the resulting cost as before, so 102.

Consequently, the price of anarchy of the new game equals $\frac{107}{102}$, while the price of stability remains 1.