Dominance Notions

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Overview

- Strict dominance.
- Weak dominance.
- Never best responses.
- Examples.

Strict and Weak Dominance

- s'_i is strictly dominated by s_i if

$$\forall s_{-i} \in S_{-i}\ p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}),$$

- s'_i is weakly dominated by s_i if

$$\forall s_{-i} \in S_{-i}\ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}),$$
$$\exists s_{-i} \in S_{-i}\ p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}).$$

Prisoner's Dilemma Reviewed

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Why a dilemma? (Another interpretation.)

- (C, C) is a unique social optimum.
- (D, D) is a unique Nash equilibrium.
- For each player C is strictly dominated by D.

Quiz

	H	T	E
H	$1, -1$	$-1, \ \ 1$	$-1, -1$
T	$-1, \ \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

- What are the Nash equilibria of this game?

Answer

	H	T	E
H	$1, -1$	$-1, \quad 1$	$-1, -1$
T	$-1, \quad 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

- (E, E) is the only Nash equilibrium.
- It is a Nash equilibrium in weakly dominated strategies.

IESDS: Example 1

	L	M	R
T	$3,0$	$2,1$	$1,0$
C	$2,1$	$1,1$	$1,0$
B	$0,1$	$0,1$	$0,0$

- B is strictly dominated by T,
- R is strictly dominated by M.

By eliminating them we get:

	L	M
T	$3,0$	$2,1$
C	$2,1$	$1,1$

IESDS, Example 1ctd

	L	M
T	$3, 0$	$2, 1$
C	$2, 1$	$1, 1$

Now C is strictly dominated by T, so we get:

	L	M
T	$3, 0$	$2, 1$

Now L is strictly dominated by M, so we get:

	M
T	$2, 1$

We solved the game by IESDS.

IESDS

Theorem

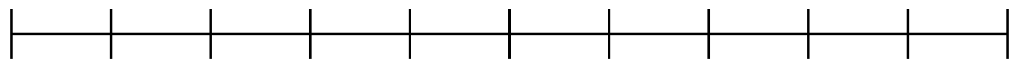
- If G' is an outcome of IESDS starting from a finite G, then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is finite and is solved by IESDS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Gilboa, Kalai, Zemel, '90) Outcome of IESDS is unique (order independence).

IESDS: Example

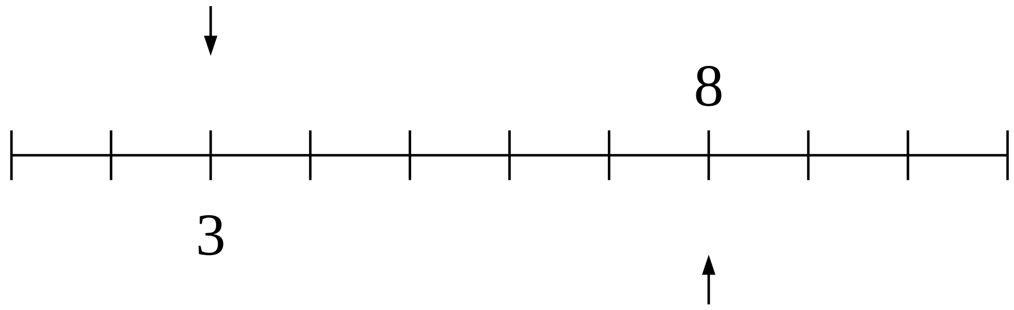
Location game (Hotelling '29)

- 2 companies decide simultaneously their location,
- customers choose the closest vendor.

Example: Two bakeries, one (discrete) street.

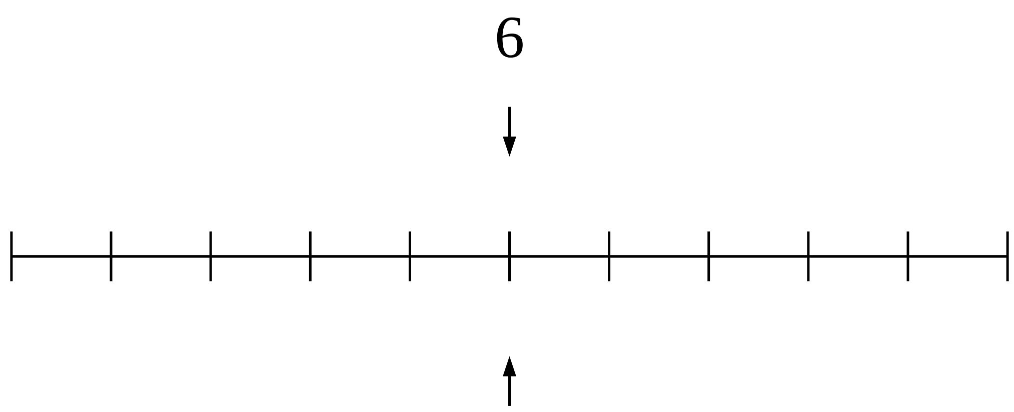


For instance:



Then baker$_1(3, 8) = 5$, baker$_2(3, 8) = 6$.
Where do I put my bakery?

Answer



Then:
baker$_1(6, 6) = 5.5$,
baker$_2(6, 6) = 5.5$.

- $(6, 6)$ is the outcome of IESDS.
- Hence $(6, 6)$ is a unique Nash equilibrium.

IEWDS

Theorem

- If G' is an outcome of IEWDS starting from a finite G and s is a Nash equilibrium of G', then s is a Nash equilibrium of G.
- If G is finite and is solved by IEWDS, then the resulting joint strategy is a Nash equilibrium of G.
- Outcome of IEWDS does not need to be unique (no order independence).

IEWDS: Beauty-contest Game

Example: The 2nd Maldives Mr & Miss Beauty Contest.

Beauty-contest Game (ctd)

[Moulin, '86]

- each set of strategies = $\{1, \ldots, 100\}$,
- payoff to each player:
  1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.

Example
submissions: $29, 32, 29$; average: 30,
payoffs: $\frac{1}{2}, 0, \frac{1}{2}$.

- This game is solved by IEWDS.
- Hence it has a Nash equilibrium, namely $(1, \ldots, 1)$.

IEWDS: Example 2

The following game has two Nash equilibria:

	X	Y	Z
A	$2,1$	$0,1$	$1,0$
B	$0,1$	$2,1$	$1,0$
C	$1,1$	$1,0$	$0,0$
D	$1,0$	$0,1$	$0,0$

- D is weakly dominated by A,
- Z is weakly dominated by X.

By eliminating them we get:

	X	Y
A	$2,1$	$0,1$
B	$0,1$	$2,1$
C	$1,1$	$1,0$

Example 2, ctd

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$
C	$1, 1$	$1, 0$

Next, we get

	X
A	$2, 1$
B	$0, 1$
C	$1, 1$

and finally

	X
A	$2, 1$

IEWDS: Example 3

	L	R
T	$1,1$	$1,1$
B	$1,1$	$0,0$

can be reduced both to

	L	R
T	$1,1$	$1,1$

and to

	L
T	$1,1$
B	$1,1$

Infinite Games

Consider the game with

- $S_i := \mathbb{N}$,
- $p_i(s) := s_i$.

Here

- every strategy is strictly dominated,
- in one step we can eliminate
  - all strategies,
  - all $\neq 0$ strategies,
  - one strategy per player.

Infinite Games (2)

Conclusions For infinite games

- IESDS is not order independent,
- definition of order independence has to be modified.

IENBR: Example 1

	X	Y
A	$2,1$	$0,0$
B	$0,1$	$2,0$
C	$1,1$	$1,2$

- No strategy strictly or weakly dominates another one.
- C is never a best response.

Eliminating it we get

	X	Y
A	$2,1$	$0,0$
B	$0,1$	$2,0$

from which in two steps we get

	X
A	$2,1$

Theorem

- If G' is an outcome of IENBR starting from a finite G, then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is finite and is solved by IENBR, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Apt, '05) Outcome of IENBR is unique (order independence).

IENBR: Example 2

Location game on the open real interval $(0, 100)$.

$$p_i(s_i, s_{3-i}) := \begin{cases} s_i + \frac{s_{3-i} - s_i}{2} & \text{if } s_i < s_{3-i} \\ 100 - s_i + \frac{s_i - s_{3-i}}{2} & \text{if } s_i > s_{3-i} \\ 50 & \text{if } s_i = s_{3-i} \end{cases}$$

- No strategy strictly or weakly dominates another one.
- Only 50 is a best response to some strategy (namely 50).
- So this game is solved by IENBR, in one step.