Social Network Games
Short Version

Sunil Simon

Centre for Mathematics and Computer Science (CWI), Amsterdam

Krzysztof R. Apt

Centre for Mathematics and Computer Science (CWI) and ILLC,
University of Amsterdam, The Netherlands

Abstract

One of the natural objectives of the field of the social networks is to predict agents' behaviour. To better understand the spread of various products through a social network [1] introduced a threshold model, in which the nodes influenced by their neighbours can adopt one out of several alternatives. To analyze the consequences of such product adoption we associate here with each such social network a natural strategic game between the agents.

In these games the payoff of each player weakly increases when more players choose his strategy, which is exactly opposite to the congestion games. The possibility of not choosing any product results in two special types of (pure) Nash equilibria.

We show that such games may have no Nash equilibrium and that determining an existence of a Nash equilibrium, also of a special type, is NP-complete. This implies the same result for a more general class of games, namely polymatrix games. The situation changes when the underlying graph of the social network is a DAG, a simple cycle, or, more generally, has no source nodes. For these three classes we determine the complexity of an existence of (a special type of) Nash equilibria.

We also clarify for these categories of games the status and the complexity of the finite best response property (FBRP) and the finite improvement property (FIP). Further, we introduce a new property of the uniform FIP which is satisfied when the underlying graph is a simple cycle, but determining it is co-NP-hard in the general case and also when the underlying graph has no source nodes. The latter complexity results also hold for the property of being a weakly acyclic game. A preliminary version of this paper appeared as [5].

Keywords: Social networks, strategic games, Nash equilibrium, finite improvement property, complexity.

1 Preliminaries

1.1 Strategic games

Assume a set $\{1, \ldots, n\}$ of players, where $n > 1$. A ***strategic game*** for n players, written as $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, consists of a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$, for each player i.

Fix a strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$. We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a ***joint strategy***, denote the ith element of s by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s.

We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if $\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. We call a joint strategy s a ***Nash equilibrium*** if each s_i is a best response to s_{-i}, that is, if

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Further, we call a strategy s'_i of player i a ***better response*** given a joint strategy s if $p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$.

Given a joint strategy s we call the sum $SW(s) = \sum_{j=1}^{n} p_j(s)$ the ***social welfare*** of s. When the social welfare of s is maximal we call s a ***social optimum***. Recall that, given a finite game that has a Nash equilibrium, its ***price of anarchy*** (respectively, ***price of stability***) is the ratio $\frac{SW(s)}{SW(s')}$ where s is a social optimum and s' is a Nash equilibrium with the lowest (respectively, highest) social welfare. In the case of division by zero, we interpret the outcome as ∞.

Following the terminology of [4], a ***path*** in S is a sequence $(s^1, s^2, \ldots)$ of joint strategies such that for every $k > 1$ there is a player i such that $s^k = (s'_i, s^{k-1}_{-i})$ for some $s'_i \neq s^{k-1}_i$. A path is called an ***improvement path*** if it is maximal and for all $k > 1$, $p_i(s^k) > p_i(s^{k-1})$ where i is the player who deviated from s^{k-1}. If an improvement path satisfies the additional property that s^k_i is a best response to s^{k-1}_{-i} for all $k > 1$ then it is called a ***best response improvement path***.

The last condition simply means that each deviating player selects a better (best) response. A game has the ***finite improvement property*** (***FIP***), (respectively, the ***finite best response property*** (***FBRP***)) if every improvement path (respectively, every best response improvement path) is finite. Obviously, if a game has the FIP or the FBRP, then it has a Nash equilibrium — it is the last element of each path. Finally, a game is called ***weakly acyclic*** (see [6, 3]) if for every joint strategy there exists a finite improvement path that starts at it.

1.2 Social networks

We are interested in specific strategic games defined over social networks. In what follows we focus on a model of the social networks recently introduced in [1].

Let $V = \{1, \ldots, n\}$ be a finite set of ***agents*** and $G = (V, E, w)$ a weighted directed graph with $w_{ij} \in [0, 1]$ being the weight of the edge (i, j). We assume that G does not have self loops, i.e., for all $i \in \{1, \ldots, n\}$, $(i, i) \notin E$. We often use the notation $i \to j$ to denote $(i, j) \in E$ and write $i \to^* j$ if there is a path from i to j in the graph G. Given a node i of G we denote by $N(i)$ the set of nodes from which there is an incoming edge to i. We call each $j \in N(i)$ a ***neighbour*** of i in G. We assume that for each node i such that $N(i) \neq \emptyset$, $\sum_{j \in N(i)} w_{ji} \leq 1$. An agent $i \in V$ is said to be a ***source node*** in G if $N(i) = \emptyset$.

Let $\mathcal{P}$ be a finite set of alternatives or ***products***. By a ***social network*** (from now on, just ***network***) we mean a tuple $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where P assigns to each agent i a non-empty set of products $P(i)$ from which it can make a choice. θ is a ***threshold function*** that for each $i \in V$ and $t \in P(i)$ yields a value $\theta(i, t) \in (0, 1]$.

Given a network $\mathcal{S}$ we denote by $\mathit{source}(\mathcal{S})$ the set of source nodes in the underlying graph G. One of the classes of the networks we shall study are the ones with $\mathit{source}(\mathcal{S}) = \emptyset$. We call a network ***equitable*** if the weights are defined as $w_{ji} = \frac{1}{|N(i)|}$ for all nodes i and $j \in N(i)$.

1.3 Social network games

Fix a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Each agent can adopt a product from his product set or choose not to adopt any product. We denote the latter choice by t_0.

With each network $\mathcal{S}$ we associate a strategic game $\mathcal{G}(\mathcal{S})$. The idea is that the nodes simultaneously choose a product or abstain from choosing any. Subsequently each node assesses his choice by comparing it with the choices made by his neighbours. Formally, we define the game as follows:

- the players are the agents,
- the set of strategies for player i is $S_i := P(i) \cup \{t_0\}$,
- for $i \in V$, $t \in P(i)$ and a joint strategy s, let $\mathcal{N}_i^t(s) := \{j \in N(i) \mid s_j = t\}$, i.e., $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted the product t in s.

  The payoff function is defined as follows, where c_0 is some positive constant given in advance:

  - for $i \in \mathit{source}(\mathcal{S})$,
  $$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ c_0 & \text{if } s_i \in P(i) \end{cases}$$
  - for $i \notin \mathit{source}(\mathcal{S})$,
  $$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ (\sum\limits_{j \in \mathcal{N}_i^t(s)} w_{ji}) - \theta(i, t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$$

Let us explain the underlying motivations behind the above definition. In the first item we assume that the payoff function for the source nodes is constant only for simplicity. In the last section of the paper we explain that the obtained results hold equally well in the case when the source nodes have arbitrary positive utility for each product.

The second item in the payoff definition is motivated by the following considerations. When agent i is not a source node, his 'satisfaction' from a joint strategy depends positively on the accumulated weight (read: 'influence') of his neighbours who made the same choice as he did, and negatively from his threshold level (read: 'resistance') to adopt this product. The assumption that $\theta(i,t) > 0$ reflects the view that there is always some resistance to adopt a product. So when this resistance is high, it can happen that the payoff is negative. Of course, in such a situation not adopting any product, represented by the strategy t_0, is a better alternative.

The presence of this possibility allows each agent to refrain from choosing a product. This refers to natural situations, such as deciding not to purchase a smartphone or not going on vacation. In the last section we refer to an initiated research on social network games in which the strategy t_0 is not present. Such games capture situations in which the agents have to take some decision, for instance selecting a secondary school for their children.

By definition the payoff of each player depends only on the strategies chosen by his neighbours, so social network games are related to graphical games of [2]. However, the underlying dependence structure of a social network game is a directed graph and the presence of the special strategy t_0 available to each player makes these games more specific. Also, these games satisfy the ***join the crowd*** property that we define as follows:

> Each payoff function p_i depends only on the strategy chosen by player i and the set of players who also chose his strategy. Moreover, the dependence on this set is monotonic.

In what follows for $t \in \mathcal{P} \cup \{t_0\}$ we use the notation $\overline{t}$ to denote the joint strategy s where $s_j = t$ for all $j \in V$. This notation is legal only if for all agents i it holds that $t \in P(i)$.

The presence of the strategy t_0 motivates the introduction and study of special types of Nash equilibria. We say that a Nash equilibrium s is

- ***determined*** if for all i, $s_i \neq t_0$,
- ***non-trivial*** if for some i, $s_i \neq t_0$,
- ***trivial*** if for all i, $s_i = t_0$, i.e., $s = \overline{t_0}$.

2 Nash equilibria: general case

The first natural question that we address is that of the existence of Nash equilibria in social network games. We have the following example.

Example 1. Consider the network given in Figure 1, where the product set of each agent is marked next to the node denoting it and the weights are labels on the edges. The source nodes are represented by the unique product in the product set.

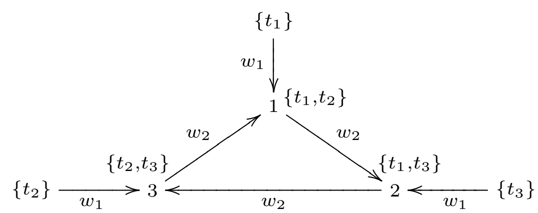


Figure 1: A network with no Nash equilibrium

So the weights on the edges from the nodes $\{t_1\}, \{t_2\}, \{t_3\}$ are marked by w_1 and the weights on the edges forming the triangle are marked by w_2. We assume that each threshold is a constant θ, where $\theta < w_1 < w_2$. So it is more profitable to a player residing on the triangle to adopt the product adopted by his neighbour residing on a triangle than by the other neighbour who is a source node. For convenience we represent each joint strategy as a triple of strategies of players 1, 2 and 3.

It is easy to check that in the game associated with this network no joint strategy is a Nash equilibrium. Indeed, each agent residing on the triangle can secure a payoff of at least $w_1 - \theta > 0$, so it suffices to analyze the joint strategies in which t_0 is not used. There are in total eight such joint strategies. Here is their listing, where in each joint strategy we underline the strategy that is not a best response to the choice of other players: $(\underline{t_1}, t_1, t_2)$, $(t_1, t_1, \underline{t_3})$, $(t_1, t_3, \underline{t_2})$, $(t_1, \underline{t_3}, t_3)$, $(t_2, \underline{t_1}, t_2)$, $(t_2, \underline{t_1}, t_3)$, $(t_2, t_3, \underline{t_2})$, $(\underline{t_2}, t_3, t_3)$. □

The social network in Example 1 used three products. The following result shows that to construct a social network $\mathcal{S}$ such that $\mathcal{G}(\mathcal{S})$ has no Nash equilibrium in fact at least three products are required.

Theorem 2. *For a network $\mathcal{S}$, if there exists a non-empty set $X \subseteq \mathcal{P}$ such that $|X| \leq 2$ and for all $i \in source(\mathcal{S})$, $P(i) \cap X \neq \emptyset$ then $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium.*

In particular $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium when all nodes i have the same set of two products.

Proof. Given an initial joint strategy we call a maximal sequence of best response deviations to a given strategy t (in an arbitrary order) a ***t-phase***. Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ where $G = (V, E, w)$.

First, suppose that $|X| = 1$, say $X = \{t_1\}$. Let s be the resulting joint strategy after performing a t_1-phase starting in the joint strategy $\overline{t_0}$. We show that s is a Nash equilibrium. First note that $s_j = t_1$ for every $j \in source(\mathcal{S})$. Further, in the t_1-phase, if a joint strategy s^2 is obtained from s^1 by having some nodes switch to product t_1 and t_1 is a best response for a node i to s^1_{-i}, then t_1 remains a best response for i to s^2_{-i}. Indeed, by the join the crowd property $p_i(t_1, s^2_{-i}) \geq p_i(t_1, s^1_{-i})$ and $p_i(t_1, s^1_{-i}) \geq p_i(t_0, s^1_{-i})$, so $p_i(t_1, s^2_{-i}) \geq p_i(t_0, s^2_{-i})$. Consequently after the first t_1-phase, in the resulting joint strategy s, each node that has the strategy t_1 plays a best response. If for some j, $s_j = t_0$ then by the definition of the t_1-phase, j is playing his best response as well. Therefore s is a Nash equilibrium.

Now suppose that $|X| = 2$, say $X = \{t_1, t_2\}$. Let $V_{t_1} = \{j \in \mathit{source}(\mathcal{S}) \mid t_1 \in P(j)\}$ and $\overline{V}_{t_1} = \{j \in \mathit{source}(\mathcal{S}) \mid t_1 \notin P(j)\}$. Let $\mathcal{S}^{t_1} = (G^{t_1}, \mathcal{P}, P, \theta)$, where G^{t_1} is the induced subgraph of G on the nodes $V \setminus \overline{V}_{t_1}$. Let s^{t_1} be the resulting joint strategy in $\mathcal{S}^{t_1}$ after performing a t_1-phase starting in $\overline{t_0}$. By the previous argument, s^{t_1} is a Nash equilibrium in $\mathcal{G}(\mathcal{S}^{t_1})$. Now consider the joint strategy s in $\mathcal{G}(\mathcal{S})$ defined as follows:

$$s_i = \begin{cases} t_0 & \text{if } i \in \overline{V}_{t_1} \\ s_i^{t_1} & \text{otherwise} \end{cases}$$

Starting at s, we repeatedly perform a t_2-phase followed by a t_0-phase. We claim that this process terminates in a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

First note that if a joint strategy s^2 is obtained from s^1 by having some nodes switch to product t_2 and t_2 is a best response for a node i to s^1_{-i}, then t_2 remains a best response for i to s^2_{-i}. The argument is analogous to the one in the previous case. Therefore after the first t_2-phase each node that has the strategy t_2 plays a best response. Call the outcome of the first t_2-phase s''.

Now consider a node i that deviated to t_0 starting at s'' by means of a best response. By the observation just made, node i deviated from product t_1. So, again by the join the crowd property, this deviation does not affect the property that the nodes that selected t_2 in s'' play a best response. Iterating this reasoning we conclude that after the first t_0-phase each node that has the strategy t_2 continues to play a best response.

By the same reasoning subsequent t_2 and t_0-phases have the same effect on the set of nodes that have the strategy t_2, namely that each of these nodes continues to play a best response.

Moreover, this set continues to weakly increase. Consequently these repeated applications of the t_2-phase followed by the t_0-phase terminate, say in a joint strategy s'. Now suppose that a node i does not play a best response to s'_{-i}. Then clearly $i \notin \mathit{source}(\mathcal{S})$. If $s'_i = t_0$, then by the construction t_2 is not a best response, so t_1 is a best response.

Suppose $s_i = t_0$. Consider the joint strategy s^{t_1} which is a Nash equilibrium in $\mathcal{G}(\mathcal{S}^{t_1})$. We have $p_i(t_1, s^{t_1}_{-i}) \leq p_i(t_0, s^{t_1}_{-i})$. Since $i \notin \mathit{source}(\mathcal{S})$, we have $s_i = s_i^{t_1}$. Since for all $j \in \overline{V}_{t_1}$, $t_1 \notin P(j)$ we have $p_i(t_1, s_{-i}) \leq p_i(t_0, s_{-i})$ as well. By the join the crowd property $p_i(t_1, s'_{-i}) \leq p_i(t_1, s_{-i})$, so $p_i(t_1, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction. Hence node i deviated to t_0 from some intermediate joint strategy s^1 by selecting a best response. So $p_i(t_1, s^1_{-i}) \leq p_i(t_0, s^1_{-i})$. Moreover, by the join the crowd property $p_i(t_1, s'_{-i}) \leq p_i(t_1, s^1_{-i})$, so $p_i(t_1, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction, as well.

Further, by the construction $s'_i \neq t_2$, so the only alternative is that $s'_i = t_1$. But then either t_0 or t_2 is a best response, which contradicts the construction of s'. We conclude that s' is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$. □

3 Nash equilibria: special cases

In view of the fact that in general Nash equilibria may not exist we now consider networks with special properties of the underlying directed graph. We focus on three natural classes.

3.1 Directed acyclic graphs

We consider first networks whose underlying graph is a directed acyclic graph (DAG). Intuitively, such networks correspond to hierarchical organizations. This restriction leads to a different outcome in the analysis of Nash equilibria.

Given a DAG $G := (V, E)$, we use a fixed level by level enumeration rank() of its nodes so that for all $i, j \in V$

$$\text{if } \mathrm{rank}(i) < \mathrm{rank}(j), \text{ then there is no path in } G \text{ from } j \text{ to } i. \tag{1}$$

Theorem 3. *Consider a network $\mathcal{S}$ whose underlying graph is a DAG. Then $\mathcal{G}(\mathcal{S})$ always has a non-trivial Nash equilibrium.*

Proof. We proceed by assigning to each node a strategy following the order determined by (1). Given a node we assign to it a best response to the sequence of strategies already assigned to all his neighbours. (By definition the strategies of other players have no influence on the choice of a best response.) This yields a non-trivial Nash equilibrium. □

Note that when the underlying graph is a DAG all Nash equilibria are non-trivial. Further, in the procedure described in the above proof, in general more than one best response can exist. In that case multiple Nash equilibria exist.

The above procedure uses the set of best responses $BR_i(s_{N(i)})$ of player i to the joint strategy $s_{N(i)}$ of his neighbours in $\mathcal{G}(\mathcal{S})$. This set is defined directly in terms of $s_{N(i)}$ and $\mathcal{S}$ as follows, where $\mathcal{S} = (G, \mathcal{P}, P, \theta)$.

Let

$$Z_i^{>0}(s_{N(i)}) := \{t \in P(i) \mid \sum_{k \in N(i), s_k = t} w_{ki} - \theta(i,t) > 0\},$$
$$Z_i^{=0}(s_{N(i)}) := \{t_0\} \cup \{t \in P(i) \mid \sum_{k \in N(i), s_k = t} w_{ki} - \theta(i,t) = 0\}.$$

Then

$$BR_i(s_{N(i)}) := \begin{cases} \underset{t \in Z_i^{>0}(s_{N(i)})}{\mathrm{argmax}} \Big(\sum_{k \in N(i), s_k = t} w_{ki} - \theta(i,t) \Big) & \text{if } Z_i^{>0}(s_{N(i)}) \neq \emptyset \\ Z_i^{=0}(s_{N(i)}) & \text{otherwise} \end{cases}$$

Finally, we consider the price of anarchy and the price of stability for the considered class of games. The following simple result holds.

Theorem 4. *The price of anarchy and the price of stability for the games associated with the networks whose underlying graph is a DAG is unbounded.*

Proof. Consider the network depicted in Figure 2.

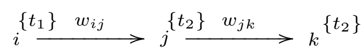


Figure 2: A network with a high price of anarchy and stability

Choose an arbitrary value $r > 0$. Suppose first that there exists weights and thresholds such that $w_{jk} - (\theta(j, t_2) + \theta(k, t_2)) > rc_0$. (Recall that the payoff of player i is c_0.)

The game associated with this network has a unique Nash equilibrium, namely the joint strategy (t_1, t_0, t_0) assigned to the sequence (i, j, k) of nodes. Its social welfare is c_0. In contrast, the social optimum is achieved by the joint strategy (t_1, t_2, t_2) and equals

$$c_0 + w_{jk} - (\theta(j, t_2) + \theta(k, t_2)) > rc_0.$$

So for every value $r > 0$ there is a network whose game has price of anarchy and price of stability higher than r.

Suppose now that the inequality $w_{jk} - (\theta(j, t_2) + \theta(k, t_2)) > rc_0$ does not hold for any choice of weights and thresholds. (This is for instance the case when $rc_0 \geq 1$, which can be the case as c_0 and r are arbitrary.) In that case, we modify the above social network as follows. First, we replace the node k by $\lceil rc_0 + 1 \rceil$ nodes, all direct descendants of node j and each with the product set $\{t_2\}$. Then we choose the weights and the thresholds in such a way that the sum of all these weights minus the sum of all the thresholds for the product t_2 exceeds rc_0. In the resulting game, by the same argument as above, both the price of anarchy and price of stability are higher than r. □

3.2 Simple cycles

Next, we consider networks whose underlying graph is a simple cycle. To fix the notation suppose that the underlying graph is $1 \to 2 \to \ldots \to n \to 1$. We assume that the counting is done in cyclic order within $\{1, \ldots, n\}$ using the increment operation $i \oplus 1$ and the decrement operation $i \ominus 1$. In particular, $n \oplus 1 = 1$ and $1 \ominus 1 = n$. The payoff functions can then be rewritten as follows:

$$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ w_{i\ominus 1\, i} - \theta(i, s_i) & \text{if } s_i = s_{i\ominus 1} \text{ and } s_i \in P(i) \\ -\theta(i, s_i) & \text{otherwise} \end{cases}$$

Clearly $\overline{t_0}$ is a trivial Nash equilibrium. The following observation clarifies when other Nash equilibria exist.

Theorem 5. *Consider a network $\mathcal{S}$ whose underlying graph is a simple cycle. Then s is a non-trivial (respectively, determined) Nash equilibrium of the game $\mathcal{G}(\mathcal{S})$ iff s is of the form $\bar{t}$ for some product t and for all i, $p_i(s) \geq 0$.*

Proof. ($\Rightarrow$) Consider a non-trival Nash equilibrium s. Suppose that $s_i = t$ for a product t. We have $p_i(s) \geq p_i(t_0, s_{-i}) = 0$, so $s_{i \ominus 1} = t$. Iterating this reasoning we conclude that $s = \bar{t}$.

($\Leftarrow$) Straightforward. □

Next, we consider the price of anarchy and the price of stability. We have the following counterpart of Theorem 4.

Theorem 6. *The price of anarchy and the price of stability for the games associated with the networks whose underlying graph is a simple cycle is unbounded.*

Proof. Choose an arbitrary value $r > 0$ and let ϵ be such that $\epsilon < min(\frac{1}{4}, \frac{1}{2(r+1)})$. Then both $1 - 2\epsilon > 2\epsilon$ and $\frac{1-2\epsilon}{2\epsilon} > r$.

Consider the network depicted in Figure 3.

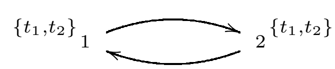


Figure 3: Another network with a high price of anarchy and stability

We assume that

$$w_{12} - \theta(2, t_2) = 1 - \epsilon,\ w_{21} - \theta(1, t_2) = -\epsilon,\ w_{12} - \theta(2, t_1) = \epsilon,\ w_{21} - \theta(1, t_1) = \epsilon.$$

Then the social optimum is achieved in the joint strategy (t_2, t_2) and equals $1 - 2\epsilon$. There are two Nash equilibria, (t_1, t_1) and the trivial one, with the respective social welfare 2ϵ and 0.

In the case of the price of anarchy we deal with the division by zero. We interpret the outcome as ∞. The price of stability equals $\frac{1-2\epsilon}{2\epsilon}$, so is higher than r. □

3.3 Graphs with no source nodes

Finally, we consider the case when the underlying graph $G = (V, E)$ of a network $\mathcal{S}$ has no source nodes, i.e., for all $i \in V$, $N(i) \neq \emptyset$. Intuitively, such a network corresponds to a 'circle of friends': everybody has a friend (a neighbour).

Here and elsewhere we only consider subgraphs that are *induced* and identify each such subgraph with its set of nodes. (Recall that (V', E') is an induced subgraph of (V, E) if $V' \subseteq V$ and $E' = E \cap (V' \times V')$.)

We say that a (non-empty) strongly connected subgraph (in short, SCS) C_t of G is ***self-sustaining*** for a product t if for all $i \in C_t$,

- $t \in P(i)$,

- $\sum_{j \in N(i) \cap C_t} w_{ji} \geq \theta(i, t)$.

An easy observation is that if $\mathcal{S}$ is a network with no source nodes, then it always has a trivial Nash equilibrium, $\overline{t_0}$. The following lemma states that for such networks every non-trivial Nash equilibrium satisfies a structural property which relates it to the set of self- sustaining SCSs in the underlying graph. We use the following notation: for a joint strategy s and a product t, $\mathcal{A}_t(s) := \{i \in V \mid s_i = t\}$ and $P(s) := \{t \mid \exists i \in V \text{ with } s_i = t\}$.

Lemma 7. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. If $s \neq \overline{t_0}$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$, then for all products $t \in P(s) \setminus \{t_0\}$ and $i \in \mathcal{A}_t(s)$ there exists a self-sustaining SCS $C_t \subseteq \mathcal{A}_t(s)$ for t and a $j \in C_t$ such that $j \to^* i$.*

Proof. Suppose $s \neq \overline{t_0}$ is a Nash equilibrium. Take any product $t \neq t_0$ and an agent i such that $s_i = t$ (by assumption, at least one such t and i exists). Recall that $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted the product t in s. Consider the set of nodes $\text{Pred} := \bigcup_{m \in \mathbb{N}} \text{Pred}_m$, where

- $\text{Pred}_0 := \{i\}$,
- $\text{Pred}_{m+1} := \text{Pred}_m \cup \bigcup_{j \in \text{Pred}_m} \mathcal{N}_j^t(s)$.

By construction for all $j \in \text{Pred}$, $s_j = t$ and $\mathcal{N}_j^t(s) \subseteq \text{Pred}$. Moreover, since s is a Nash equilibrium, we also have $\sum_{k \in \mathcal{N}_j^t(s)} w_{kj} \geq \theta(j, t)$.

Consider the partial ordering $<$ between the strongly connected components of the graph induced by Pred defined by: $C < C'$ iff $j \to k$ for some $j \in C$ and $k \in C'$. Now take some SCS C_t induced by a strongly connected component that is minimal in the $<$ ordering. Then for all $k \in C_t$ we have $\mathcal{N}_k^t(s) \subseteq C_t$ and hence $\mathcal{N}_k^t(s) \subseteq N(k) \cap C_t$. This shows that C_t is self-sustaining.

Moreover, by the construction of Pred for all $j \in \text{Pred}$, and a fortiori for all $j \in C_t$, we also have $j \to^* i$. Since the choice of t and i was arbitrary, the claim follows. □

Using Lemma 7, we can now provide a necessary and sufficient condition for the existence of non-trivial Nash equilibria for the considered networks.

Theorem 8. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. The joint strategy $\overline{t_0}$ is a unique Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there does not exist a product t and a self-sustaining SCS C_t for t in G.*

Proof. ($\Leftarrow$) By Lemma 7.
($\Rightarrow$) Suppose there exists a self-sustaining SCS C_t for a product t. Let R be the set of nodes reachable from C_t which eventually can adopt product t. Formally, $R := \bigcup_{m \in \mathbb{N}} R_m$ where

- $R_0 := C_t$,

- $R_{m+1} := R_m \cup \{j \mid t \in P(j)$ and $\sum\limits_{k \in N(j) \cap R_m} w_{kj} \geq \theta(j,t)\}$.

Let s be the joint strategy such that for all $j \in R$, we have $s_j = t$ and for all $k \in V \setminus R$, we have $s_k = t_0$. It follows directly from the definition of R that s satisfies the following properties:

(P1) for all $i \in V$, $s_i = t_0$ or $s_i = t$,

(P2) for all $i \in V$, $s_i \neq t_0$ iff $i \in R$,

(P3) for all $i \in V$, if $i \in R$ then $p_i(s) \geq 0$.

We show that s is a Nash equilibrium. Consider first any j such that $s_j = t$ (so $s_j \neq t_0$). By (P2) $j \in R$ and by (P3) $p_j(s) \geq 0$. Since $p_j(s_{-j}, t_0) = 0 \leq p_j(s)$, player j does not gain by deviating to t_0. Further, by (P1), for all $k \in N(j)$, $s_k = t$ or $s_k = t_0$ and therefore for all products $t' \neq t$ we have $p_j(s_{-j}, t') < 0 \leq p_j(s)$. Thus player j does not gain by deviating to any product $t' \neq t$ either.

Next, consider any j such that $s_j = t_0$. We have $p_j(s) = 0$ and from (P2) it follows that $j \notin R$. By the definition of R we have $\sum\limits_{k \in N(j) \cap R} w_{kj} < \theta(j,t)$. Thus $p_j(s_{-j}, t) < 0$. Moreover, for all products $t' \neq t$ we also have $p_j(s_{-j}, t') < 0$ for the same reason as above. So player j does not gain by a unilateral deviation. We conclude that s is a Nash equilibrium. □

Next, for a product $t \in \mathcal{P}$, we define the set $X_t := \bigcap_{m \in \mathbb{N}} X_t^m$, where

- $X_t^0 := \{i \in V \mid t \in P(i)\}$,
- $X_t^{m+1} := \{i \in V \mid \sum_{j \in N(i) \cap X_t^m} w_{ji} \geq \theta(i,t)\}$.

We also have the following characterization result.

Theorem 9. *Let $\mathcal{S}$ be a network whose underlying graph has no source nodes. There exists a non-trivial Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there exists a product t such that $X_t \neq \emptyset$.*

Proof. Suppose $\mathcal{S} = (G, \mathcal{P}, P, \theta)$.
($\Rightarrow$) It follows directly from the definitions that if there is a self-sustaining SCS C_t for product t, then $C_t \subseteq X_t$. Suppose now that for all t, $X_t = \emptyset$. Then for all t, there is no self-sustaining SCS for t. So by Theorem 8, $\overline{t_0}$ is a unique Nash equilibrium.
($\Leftarrow$) Suppose there exists t such that $X_t \neq \emptyset$. Let s be the joint strategy defined as follows:

$$s_i := \begin{cases} t & \text{if } i \in X_t \\ t_0 & \text{if } i \notin X_t \end{cases}$$

By the definition of X_t, for all $i \in X_t$, $p_i(s) \geq 0$. So no player $i \in X_t$ gains by deviating to t_0 (as then his payoff would become 0) or to a product $t' \neq t$ (as then his payoff would become negative since no player adopted t'). Also, by the definition of X_t and of the joint strategy s, for all $i \notin X_t$ and for all $t' \in P(i)$, $p_i(t', s_{-i}) < 0$. Therefore, no player $i \notin X_t$ gains by deviating to a product t' either. It follows that s is a Nash equilibrium. $\square$

4 Finite best response improvement property: special cases

As in the case of Nash equilibria we now analyze the FBRP for social network games whose underlying graph satisfies certain properties.

4.1 Directed acyclic graphs

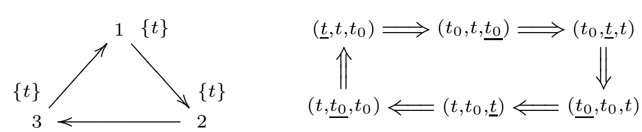
We begin with social network games whose underlying graph is a DAG. Then the following positive result holds.

Theorem 10. *Consider a network $\mathcal{S}$ whose underlying graph is a DAG. Then the game $\mathcal{G}(\mathcal{S})$ has the FBRP.*

This is a direct consequence of a stronger result, Theorem 13 of Section 5.1.

4.2 Simple cycles

The property that the game has the FBRP does not hold anymore when the underlying graph is a simple cycle. To see this consider Figure 4(a). Suppose that $\bar{t}$ is a Nash equilibrium in which each player gets a strictly positive payoff. Figure 4(b) then shows an infinite best response improvement path. In each strategy profile, we underline the strategy that is not a best response to the choice of other players.



(a) (b)

Figure 4: A network with an infinite best response improvement path

Next consider Figure 5(a). Suppose that both $\overline{t_1}$ and $\overline{t_2}$ are Nash equilibria. Then Figure 5(b) shows an infinite best response improvement path.

One can easily generalize the above two examples to simple cycles with more than three nodes. We now show that when the game does not have the FBRP, the network is necessarily of one of the above two types.

Theorem 11. *Let $\mathcal{S}$ be a network whose underlying graph is a simple cycle.*

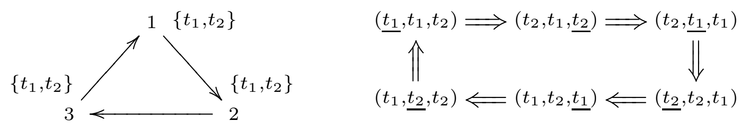


Figure 5: Another network with an infinite best response improvement path

(i) Suppose that $\mathcal{S}$ has 2 nodes. Then the game $\mathcal{G}(\mathcal{S})$ has the FBRP.

(ii) Suppose that $\mathcal{S}$ has at least 3 nodes. Then the game $\mathcal{G}(\mathcal{S})$ does not have the FBRP iff either it has a determined Nash equilibrium s such that for all i, $p_i(s) > 0$ or it has two determined Nash equilibria.

Proof. (i) A simple analysis, which we leave to the reader, shows that the longest possible improvement path is of length five and is of the form (t_1, t_2), (t_0, t_2), (t_2, t_2), (t_2, t_0), (t_0, t_0).

(ii) $(\Rightarrow)$ Consider an infinite best response improvement path ξ. Some node changes his strategy in ξ infinitely often. This means that some node, say i, selects in ξ some product t infinitely often. Indeed, otherwise from some moment on in each strategy profile in ξ its strategy would be t_0, which is not the case.

Each time node i switches in ξ to the product t, it selects a best response, so its payoff becomes at least 0. Consequently, at the moment of such a switch its predecessor $i \ominus 1$'s strategy is necessarily t as well. So if from some moment on node $i \ominus 1$ does not switch from the strategy t, then node i does not switch from t either. This shows that node $i \ominus 1$ also selects in ξ product t infinitely often. Iterating this reasoning we conclude that each node selects in ξ the product t infinitely often. Therefore, for all i, $t \in P(i)$. Since the payoff of i depends only on the choice of $i \ominus 1$, we also have that $p_i(\bar{t}) \geq 0$ for all i. By Theorem 5, $\bar{t}$ is a Nash equilibrium.

This shows that if a node selects in ξ some product t_1 infinitely often, then all nodes select in ξ the product t_1 infinitely often and $\overline{t_1}$ is a Nash equilibrium. Suppose now that $\bar{t}$ is a unique determined Nash equilibrium. This means that all other products are selected in ξ finitely often. So from some moment on in ξ nodes select only t or t_0. In this suffix η of ξ each node selects t infinitely often. Further, each switch to t from t_0 is a better response. Hence each time a node switches in η to t its payoff becomes > 0. This shows that $\bar{t}$ is a determined Nash equilibrium such that for all i, $p_i(\bar{t}) > 0$.

$(\Leftarrow)$ Suppose first that the game $\mathcal{G}(\mathcal{S})$ has a determined Nash equilibrium s such that for all i, $p_i(s) > 0$. By Theorem 5 s is of the form $\bar{t}$ for some product t. Then consider the following strategy profile:

$$s := (t, \ldots, t, t_0).$$

First schedule node 1 that has a better response, namely t_0. Next, schedule node n for which t is a best response. After these two steps the strategy profile becomes a rotation of s by 1. Iterating this selection procedure we obtain an infinite best response improvement path.

Next, suppose that the game $\mathcal{G}(\mathcal{S})$ has two determined Nash equilibria. By Theorem 5 they are of the form $\overline{t_1}$ and $\overline{t_2}$ for some products t_1 and t_2. Then consider the following strategy profile:

$$s := (t_1, \ldots, t_1, t_2).$$

First schedule node 1 for which t_2 is a best response. Next, schedule node n for which t_1 is a best response. After these two steps the strategy profile becomes a rotation of s by 1. Iterating this selection procedure we obtain an infinite best response improvement path. □

5 Finite improvement property: special cases

In this section we clarify whether the special classes of social network games have the FIP.

In what follows we make use of the following simple observation.

Note 12. *Consider a game $\mathcal{G}(\mathcal{S})$. If a node i is infinitely often selected in an improvement path, then so is a node $j \in N(i)$.* □

5.1 Graphs with special strongly connected components

We begin with the strategic games associated with the networks whose underlying graph is a DAG. Then the following positive result holds.

Theorem 13. *Consider a network $\mathcal{S}$ whose underlying graph is a DAG. Then the game $\mathcal{G}(\mathcal{S})$ has the FIP.*

Proof. Suppose not. Then by repeatedly using Note 12 we obtain an infinite path in the underlying graph. This yields a contradiction. □

We now generalize this result to a larger class of directed graphs. First we consider the case of two player games.

Theorem 14. *Every two player social network game has the FIP.*

Proof. By Theorem 13 we can assume that the underlying graph is a cycle, say $1 \to 2 \to 1$. Consider an improvement path ρ. By removing, if necessary, some steps we can assume that the players alternate their moves in ρ.

In what follows given an element of ρ (that is not the last one) we underline the strategy of the player who moves, i.e., selects a better response. We call each element of ρ of the type $(\underline{t}, t)$ or $(t, \underline{t})$ a *match* and use $\Rightarrow$ to denote the transition between two consecutive joint strategies in ρ. Further, we shorten the statement "each time player i switches his strategy his payoff strictly increases and it never decreases when his opponent switches strategy" to "player i's payoff steadily goes up".

Consider now two successive matches in ρ, based respectively on the strategies t and t_1. The corresponding segment of ρ is one of the following four types.

Type 1. $(\underline{t}, t) \Rightarrow^* (\underline{t_1}, t_1)$.

The fragment of ρ that starts at $(\underline{t}, t)$ and finishes at $(\underline{t_1}, t_1)$ has the following form:

$$(\underline{t}, t) \Rightarrow (t_2, \underline{t}) \Rightarrow^* (t_1, \underline{t_3}) \Rightarrow (\underline{t_1}, t_1).$$

Note that player 1's payoff can drop in a segment of ρ only if this segment contains a transition of the form $(t', \underline{t'}) \Rightarrow (\underline{t'}, t_1)$. So in the considered segment player 1's payoff steadily goes up. Additionally, in the step $(t_1, \underline{t_3}) \Rightarrow (\underline{t_1}, t_1)$ his payoff increases by w_{21}.

In turn, in the step $(\underline{t}, t) \Rightarrow (t_2, \underline{t})$ player 2's payoff decreases by w_{12} and in the remaining steps his payoff steadily goes up. So $p_1(\bar{t}) + w_{21} < p_1(\overline{t_1})$ and $p_2(\bar{t}) - w_{12} < p_2(\overline{t_1})$.

Type 2. $(\underline{t}, t) \Rightarrow^* (t_1, \underline{t_1})$.

For the analogous reason as above player 1's payoff steadily goes up. In turn, in the first step of $(\underline{t}, t) \Rightarrow^* (t_1, \underline{t_1})$ the payoff of player 2 decreases by w_{12}, while in the last step (in which player 1 moves) his payoff increases by w_{12}. So these two payoff changes cancel each other. Additionally, in the remaining steps player 2's payoff steadily goes up. So $p_1(\bar{t}) < p_1(\overline{t_1})$ and $p_2(\bar{t}) < p_2(\overline{t_1})$.

Type 3. $(t, \underline{t}) \Rightarrow^* (\underline{t_1}, t_1)$.

This type is symmetric to Type 2, so $p_1(\bar{t}) < p_1(\overline{t_1})$ and $p_2(\bar{t}) < p_2(\overline{t_1})$.

Type 4. $(t, \underline{t}) \Rightarrow^* (t_1, \underline{t_1})$.

This type is symmetric to Type 1, so $p_1(\bar{t}) - w_{21} < p_1(\overline{t_1})$ and $p_2(\bar{t}) + w_{12} < p_2(\overline{t_1})$.

We summarize in Table 1 the changes in the payoffs p_1 and p_2 between the considered two matches.

Type	p_1	p_2
1	increases by $> w_{21}$	decreases by $< w_{12}$
2, 3	increases	increases
4	decreases by $< w_{21}$	increases by $> w_{12}$

Table 1: Changes in p_1 and p_2

Consider now a match $(\underline{t}, t)$ in ρ and a match $(\underline{t_1}, t_1)$ that appears later in ρ. Let T_i denote the number of internal segments of type i that occur in the fragment of ρ that starts with $(\underline{t}, t)$ and ends with $(\underline{t_1}, t_1)$.

Case 1. $T_1 \geq T_4$.

Then Table 1 shows that the aggregate increase in p_1 in segments of type 1 exceeds the aggregate decrease in segments of type 4. So $p_1(\bar{t}) < p_1(\overline{t_1})$.

Case 2. $T_1 < T_4$.

Then analogously Table 1 shows that $p_2(\bar{t}) < p_2(\overline{t_1})$.

We conclude that $t \neq t_1$. By symmetry the same conclusion holds if the considered matches are of the form $(t, \underline{t})$ and $(t_1, \underline{t_1})$. This proves that each match occurs in ρ at most once. So in some suffix η of ρ no match occurs. But each step in η increases the social welfare, so η is finite, and consequently ρ is. □

In social network games the players share at least one strategy, t_0, that ensures each of them the zero payoff. Also, the weights and thresholds are drawn from specific intervals. However, these properties are not used in the above proof. As a result the above proof shows that each of the following two player games has the FIP.

- The set of strategies of player i is a finite set S_i,
- the payoff function is defined by $p_i(s) := f_i(s_i) + a_i(s_i = s_{-i})$, where $f_i : S_i \to \mathbb{R}$, $a_i > 0$ and $(s_i = s_{-i})$ is defined by

$$(s_i = s_{-i}) := \begin{cases} 1 & \text{if } s_i = s_{-i} \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, a_i can be viewed as a bonus for player i for coordinating with his opponent.

We can now draw a conclusion about a larger class of social network games.

Theorem 15. *Consider a network $\mathcal{S}$ such that each strongly connected component of the underlying graph is a cycle of length 2. Then the game $\mathcal{G}(\mathcal{S})$ has the FIP.*

Proof. Suppose $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Consider the condensation of G, i.e., the DAG G' resulting from contracting each cycle of G to a single node. We enumerate the nodes of G using the function rank() (introduced in Subsection 3.1) such that if $\text{rank}(i) < \text{rank}(j)$, then there is no path in G from j to i, and subsequently modify it to an enumeration $\text{rank}'()$ of the nodes of G', by replacing each contraction of a cycle by its two nodes.

Suppose now that an infinite improvement path ρ in $\mathcal{G}(\mathcal{S})$ exists. Choose the first node i in the enumeration $\text{rank}'()$ that is infinitely often selected in ρ. By Note 12 i lies on a cycle, say $i \to i' \to i$, in G. Consider a suffix of ρ in which the nodes that precede i in $\text{rank}'()$ do not appear anymore. In particular, by the choice of i, the neighbours of i or i' precede i in $\text{rank}'()$, so they do not appear in this suffix either. Delete from each element of this suffix the strategies of the nodes that differ from i and i'. We obtain in this way an infinite improvement path in the two player game G' associated with the weighted directed graph related to the cycle $i \to i' \to i$, which contradicts Theorem 14.

There is a small subtlety in the last step that we should clarify. The payoff functions in the game G' need to take into account the weights of the edges from all the neighbours of i and i' and the final strategies chosen by these neighbours. For instance, in the case of the directed graph from Figure 6 the game G' has the players i and i' but the weights of the edges $k \to i$ and $k \to i'$ and the final strategy of node k need to be taken into account in the computation of the payoff functions for, respectively, nodes i and i'.

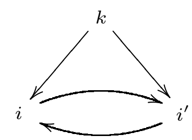


Figure 6: A directed graph

So the game G' is not exactly the social network game associated with the weighted directed graph related to $i \to i' \to i$. However, the observation stated after the proof of Theorem 14 allows us to conclude that G' does have the FIP. □

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