Mechanism Design

Krzysztof R. Apt

CWI, Amsterdam, the Netherlands,

University of Amsterdam

Overview

- Decision problems.
- Direct mechanisms.
- Groves mechanisms.
- Examples.
- Optimality results.

Intelligent Design

Economist.com

Search | Economist.com | Go

Requires subscription

Log in: e-mail | Password | Go

Remember me | Register

Finance & Economics

Economics focus

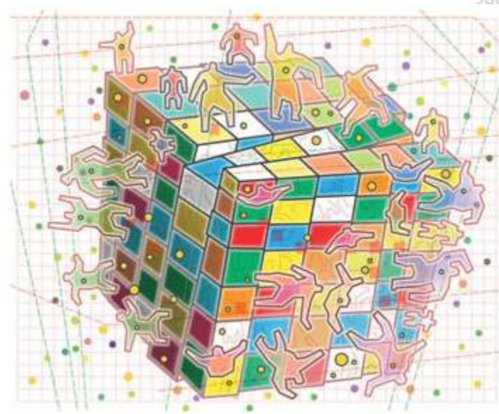
Intelligent design

Oct 18th 2007

From *The Economist* print edition

A theory of an intelligently guided invisible hand wins the Nobel prize

Jac



"WHAT on earth is mechanism design?" was the typical reaction to this year's Nobel prize in economics, announced on October 15th. In this era of "Freakonomics", in which everyone is discovering their inner economist, economics has become unexpectedly sexy. So what possessed the Nobel committee to honour a subject that sounds so thoroughly dismal? Why didn't they follow the lead of the peace-prize judges, who know not to let technicalities about being true to the meaning of the award get in the way of good headlines?

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like. The word "mechanism" refers to the institutions and the rules of the game that govern our economic activities, which can range from a Ministry of Planning in a command economy to the internal organisation of a company to trading in a market.

Intelligent Design

A theory of an intelligently guided invisible hand wins the Nobel prize

WHAT on earth is mechanism design? was the typical reaction to this year's Nobel prize in economics, announced on October 15th.

[...]

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like.

(The Economist, Oct. 18th, 2007)

Decision Problems

Decision problem for n players:

- set D of decisions,
- for each player i a set of (private) types Θ_i
- and a utility function

$$v_i : D \times \Theta_i \to \mathcal{R}.$$

- Intuitions
  - Type is some private information known only to the player (e.g., player's valuation of the item for sale),
  - $v_i(d, \theta_i)$ represents the benefit to player i of type θ_i from the decision $d \in D$.
- Assume the individual types are $\theta_1, \ldots, \theta_n$. Then $\sum_{i=1}^{n} v_i(d, \theta_i)$ is the social welfare from $d \in D$.

Decision Rules

- Decision rule is a function

$$f : \Theta_1 \times \ldots \times \Theta_n \to D.$$

- Decision rule f is efficient if

$$\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d, \theta_i)$$

  for all $\theta \in \Theta$ and $d \in D$.

- Intuition f is efficient if it always maximizes the social welfare.

Set up

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ_i',
- the central authority computes decision

$$d := f(\theta_1', \ldots, \theta_n'),$$

and communicates it to each player i.

Basic problem How to ensure that $\theta_i' = \theta_i$.

Example 1: Sealed-Bid Auction

Set up There is a single object for sale. Each player is a buyer. The decision is taken by means of a sealed-bid auction. The object is sold to the highest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- Let argsmax $\theta := \mu i(\theta_i = \max_{j \in \{1,\ldots,n\}} \theta_j)$.
- $f(\theta) :=$ argsmax θ.
- Note f is efficient.
- Payments will be treated later.

Example 2: Public Project Problem

```
Each person is asked to report his or her
willingness to pay for the project, and
the project is undertaken if and only if
the aggregate reported willingness to pay
exceeds the cost of the project.
```

(15 October 2007, The Royal Swedish Academy of Sciences, Press Release, Scientific Background)

Public Project Problem, formally

- c: cost of the public project (e.g., building a bridge),
- $D = \{0, 1\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$
- Note f is efficient.

Ex. 3: Reversed Sealed-bid Auction

Set up Each player offers the same service. The decision is taken by means of a sealed-bid auction. The service is purchased from the lowest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_-$;
  $-\theta_i$ is the price player i offers,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- $f(\theta) :=$ argsmax θ.

Example $f(-8, -5, -4, -6) = 3$. That is, given the offers $8, 5, 4, 6$, the service is bought from player 3.

Ex. 4: Buying a Path in a Network

Set up Given a graph $G := (V, E)$.

- Each edge $e \in E$ is owned by player e.
- Two distinguished vertices $s, t \in V$.
- Each player e submits the cost θ_e of using the edge e.
- The central authority selects the shortest $s - t$ path in G.

- $D = \{p \mid p$ is a $s - t$ path in $G\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(p, \theta_i) := \begin{cases} -\theta_i & \text{if } i \in p \\ 0 & \text{otherwise} \end{cases}$
- $f(\theta) := p$, where p is the shortest $s - t$ path in G.

Manipulations

Example An optimal strategy for player i in public project problem:

- if $\theta_i \geq \frac{c}{n}$ submit $\theta'_i = c$.
- if $\theta_i < \frac{c}{n}$ submit $\theta'_i = 0$.

For example, assume $c = 30$.

player	type
A	6
B	7
C	25

Players A and B should submit 0. Player c should submit 30.

Revised Set-up: Direct Mechanisms

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i; this yields $\theta' := (\theta'_1, \ldots, \theta'_n)$,
- the central authority computes decision

  $$d := f(\theta'),$$

  and taxes

  $$t(\theta') := (t_1(\theta'), \ldots, t_n(\theta')) \in \mathbb{R}^n,$$

  and communicates to each player i both d and $t_i(\theta')$.
- final utility function for player i: $u_i : D \times \mathbb{R}^n \times \Theta_i \to \mathbb{R}$ defined by

  $$u_i((f,t)(\theta), \theta_i) := v_i(f(\theta), \theta_i) + t_i(\theta).$$

Direct Mechanisms, ctd

- When the received (true) type of player i is θ_i and his announced type is θ_i', his final utility is

$$u_i((f,t)(\theta_i', \theta_{-i}), \theta_i) = v_i(f(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i}),$$

  where θ_{-i} are the types announced by the other players.

- Direct mechanism (f,t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1,\ldots,n\}$ and $\theta_i' \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f,t)(\theta_i', \theta_{-i}), \theta_i).$$

- Intuition Submitting false type (so $\theta_i' \neq \theta_i$) does not pay off.
- Direct mechanism (f,t) is feasible if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ.
- Intuition External financing is never needed.

Groves Mechanisms

- $t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) + h_i(\theta_{-i})$, where

  $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function.

- Note

  $$u_i((f, t)(\theta), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

- Intuitions
  - Player i cannot manipulate the value of $h_i(\theta_{-i})$.
  - Suppose $h_i(\theta_{-i}) = 0$.
    When the individual types are $\theta_1, \ldots, \theta_n$
    $u_i((f, t)(\theta), \theta_i)$ is the social welfare from decision $f(\theta)$.

Groves Theorem

Theorem (Groves '73)
Suppose f is efficient. Then each Groves mechanism is incentive compatible.

Proof.
For all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$\begin{aligned} u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) &= \sum_{j=1}^{n} v_j(f(\theta_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i}) \\ (f \text{ is efficient}) &\geq \sum_{j=1}^{n} v_j(f(\theta'_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i}) \\ &= u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i). \end{aligned}$$

Special Case: Pivotal Mechanism

- $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j)$.
- Then

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j) \leq 0.$$

- Note Pivotal mechanism is feasible.

Re: Sealed-Bid Auction

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\, \theta. \\ 0 & \text{otherwise} \end{cases}$$

So the pivotal mechanism is Vickrey auction (Vickrey '61):
the winner pays the 2nd highest bid.

Example

player	bid	tax to authority	util.
A	18	0	0
B	24	-21	3
C	21	0	0

Social welfare: 0 + 0 + 3 = 3.

Maximizing Social Welfare

Question: Does Vickrey auction maximize social welfare?

Notation θ^*: the reordering of θ is descending order.

Example For $\theta = (1, 4, 2, 3, 1)$ we have
$\theta_{-2} = (1, 2, 3, 0)$,
$(\theta_{-2})^* = (3, 2, 1, 0)$,
so $(\theta_{-2})^*_2 = 2$.

Intuition $(\theta_{-2})^*_2$ is the second highest bid among other bids.

Bailey-Cavallo Mechanism

(Bailey '97, Cavallo '06)

Assume $n \geq 3$.

$$t_i(\theta) := t_i^p(\theta) + \frac{(\theta_{-i})_2^*}{n}$$

Note Bailey-Cavallo mechanism is a Groves mechanism.

Example

player	bid	tax to authority	util.	why?
A	18	0	7	(= $1/3$ of 21)
B	24	-2	9	(= 3 + ($1/3$ of 18))
C	21	0	6	(= $1/3$ of 18)

Bailey-Cavallo Mechanism, ctd

Note Bailey-Cavallo mechanism is feasible.

Proof. For all i and θ, $(\theta_{-i})^*_2 \leq \theta^*_2$, so

$$\sum_{i=1}^{n} t_i(\theta) = -\theta^*_2 + \sum_{i=1}^{n} \frac{(\theta_{-i})^*_2}{n} = \sum_{i=1}^{n} \frac{-\theta^*_2 + (\theta_{-i})^*_2}{n} \leq 0.$$

Bailey-Cavallo mechanism is not an auction, because the losers may receive a payment.

Re: Public Project Problem

Assume the pivotal mechanism.
Examples Suppose $c = 30$ and $n = 3$.

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

Social welfare can be negative.

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	22	0	0

Formally

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} 0 & \text{if } \sum_{j \neq i} \theta_j \geq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ \sum_{j \neq i} \theta_j - \frac{n-1}{n} c & \text{if } \sum_{j \neq i} \theta_j < \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ 0 & \text{if } \sum_{j \neq i} \theta_j \leq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \\ \frac{n-1}{n} c - \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j > \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \end{cases}$$

This is the mechanism essentially proposed in Clarke '71).

Optimality Result (1)

Theorem (Apt, Conitzer, Guo and Markakis ’08)
Consider the sealed bid auction.
No tax-based mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Bailey-Cavallo mechanism.

Optimality Result (2)

Theorem (Apt, Conitzer, Guo and Markakis ’08)
Consider the public project problem.
No tax-based mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Clarke’s tax.

However . . .

Pivotal mechanism is not optimal in the public project problem

- when the payments per player can differ.

Re: Reversed Sealed-Bid Auction

Take

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D \setminus \{i\}} \sum_{j \neq i} v_j(d, \theta_j).$$

Note

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\, \theta. \\ 0 & \text{otherwise} \end{cases}$$

So in this mechanism the winner receives the amount equal to the 2nd lowest offer.

Example Consider $\Theta = (-8, -5, -4, -6)$. The service is bought from player 3 who receives for it 5.

Re: Buying a Path in a Network

(Nisan, Ronen '99)
Take

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{p \in D(G \setminus \{i\})} \sum_{j \neq i} v_j(p, \theta_j).$$

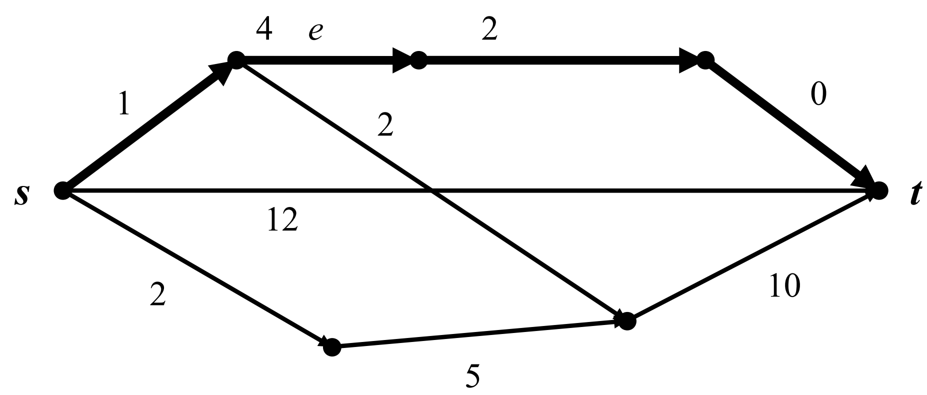
Note

$$t_i(\theta) = \begin{cases} cost(p_2) - cost(p_1 - \{i\}) & \text{if } i \in p_1 \\ 0 & \text{otherwise} \end{cases}$$

where

p_1 is the shortest $s - t$ path in $G(\theta)$,
p_2 is the shortest $s - t$ path in $(G \setminus \{i\})(\theta_{-i})$.

Example



Consider the player owning the edge e.
To compute the payment he receives

- determine the shortest $s - t$ path. Its length is 7. It contains e.
- determine the shortest $s - t$ path that does not include e. Its length is 12.
- So player e receives $12 - (7 - 4) = 9$.
  His final utility is $9 - 4 = 5$.

Pre-Bayesian Games

Pre-Bayesian Games

(Hyafil, Boutilier '04, Ashlagi, Monderer, Tennenholtz '06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (complete information).
- In a pre-Bayesian game after each player selected his strategy each player knows only his payoff (incomplete information).
- This is achieved by introducing (private) types.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \geq 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i : A_1 \times \ldots \times A_n \times \Theta_i \to \mathbb{R}$,

for each player i.

Basic assumptions:

- Nature moves first and provides each player i with a θ_i,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Ex-post Equilibrium

- A strategy for player i:

$$s_i(\cdot) \in A_i^{\Theta_i}.$$

- Joint strategy $s_(\cdot)$ is an ex-post equilibrium if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$:

$$\forall \theta \in \Theta \ \forall i \in \{1, \ldots, n\} \ \forall s_i'(\cdot) \in A_i^{\Theta_i}$$
$$p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Note: For each $\theta \in \Theta$ we have one strategic game.
$s(\cdot)$ is an ex-post equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is a Nash equilibrium in the θ-game.

Quiz

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

U, L

	F	B
F	2, 1	2, 0
B	0, 1	2, 1

U, R

	F	B
F	2, 0	2, 1
B	0, 0	2, 1

D, L

	F	B
F	3, 1	2, 0
B	5, 1	4, 1

D, R

	F	B
F	3, 0	2, 1
B	5, 0	4, 1

Which strategies form an ex-post equilibrium?

Answer

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

U, L:

	F	B
F	$2, 1$	$2, 0$
B	$0, 1$	$2, 1$

U, R:

	F	B
F	$2, 0$	$2, 1$
B	$0, 0$	$2, 1$

D, L:

	F	B
F	$3, 1$	$2, 0$
B	$5, 1$	$4, 1$

D, R:

	F	B
F	$3, 0$	$2, 1$
B	$5, 0$	$4, 1$

- Strategies
  $s_1(U) = F, s_1(D) = B$,
  $s_2(L) = F, s_2(R) = B$
  form an ex-post equilibrium.

But . . .

Ex-post equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

- $\Theta_1 = \{U, B\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{C, D\}$.

L:

		C	D
U	C	$2, 2$	$0, 0$
	D	$3, 0$	$1, 1$
B	C	$1, 2$	$3, 0$
	D	$0, 0$	$2, 1$

R:

		C	D
U	C	$2, 1$	$0, 0$
	D	$3, 0$	$1, 2$
B	C	$1, 1$	$3, 0$
	D	$0, 0$	$2, 2$

Safety-level Equilibrium

- Strategy $s_i(\cdot)$ for player i is a safety-level best response to $s_{-i}(\cdot)$ if for all strategies $s'_i(\cdot)$ of player i and all $\theta_i \in \Theta_i$

$$\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq \min_{\theta_{-i}\in\Theta_{-i}} p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Intuition $\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$ is the guaranteed payoff to player i when his type is θ_i and $s(\cdot)$ are the selected strategies.
- Joint strategy $s(\cdot)$ is a safety-level equilibrium if each $s_i(\cdot)$ is a safety-level best response to $s_{-i}(\cdot)$.
- Theorem (Ashlagi, Monderer, Tennenholtz '06)
  Every mixed extension of a finite pre-Bayesian game has a safety-level equilibrium.

Relation to Mechanism Design

- Strategy $s_i(\cdot)$ is dominant if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \; p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a revelation-type if $A_i = \Theta_i$ for all $i \in \{1, \ldots, n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i.
- A strategy for player i is called truth-telling if it is the identity function $\pi_i(\cdot)$.

Relation to Mechanism Design, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
  - Each Θ_i as in the mechanism,
  - Each $A_i = \Theta_i$,
  - $p_i(\theta'_i, \theta_{-i}, \theta_i) := u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i)$.
- Note Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- Conclusion In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_i(\cdot))$ is a dominant strategy ex-post equilibrium.