Potential Games

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Overview

- Best response dynamics.
- Potentials.
- Congestion games.
- Fair cost sharing games.
- Braess Paradox.
- Price of Stability.

Best Response Dynamics

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- Best response dynamics:
  an algorithm to find a Nash equilibrium:

  choose $s \in S_1 \times \cdots \times S_n$;
  while s is not a NE **do**
     choose $i \in \{1, \ldots, n\}$ such that
        s_i is not a best response to s_{-i};
     $s_i :=$ a best response to s_{-i}
  od

Best Response Dynamics, ctd

Note Assume a game for n players.
Suppose every player has a strictly dominant strategy. Then all best response dynamics terminate after at most n steps and their outcomes is unique.

Best Response Dynamics, ctd

Note Best response dynamics may miss a Nash equilibrium.

Example

	H	T	E
H	$1, -1$	$-1, \ \ 1$	$-1, -1$
T	$-1, \ \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

Potentials

(Monderer and Shapley '96)

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- Function $P : S_1 \times \cdots \times S_n \to \mathbb{R}$ is a potential function for G if

$$\forall i \in \{1, \ldots, n\} \; \forall s_{-i} \in S_{-i} \; \forall s_i, s_i' \in S_i$$
$$p_i(s_i, s_{-i}) - p_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}).$$

- Intuition: P tracks the changes in the payoff when some player deviates.
- Potential game: a game that has a potential function.

Example 1

- Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

- Potential

	C	D
C	0	1
D	1	2

- Intuition: potential counts the number of defecting players.

Example 2

- Prisoner's dilemma for n players.

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

  1 (formerly C),
  0 (formerly D).

- For $i = 1, 2$

$$p_i(0, s_{-i}) - p_i(1, s_{-i}) = 1.$$

- So $P(s) := n - \sum_{j=1}^{n} s_j$ is a potential function.
- Intuition: potential counts the number of defecting players.

Potential Games

Note For finite potential games all best response dynamics terminate.

Proof. Along each best response path the potential strictly increases.

Ordinal Potentials

Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- Function $P : S_1 \times \cdots \times S_n \rightarrow \mathbb{R}$ is an ordinal potential for G if

$$\forall i \in \{1, \ldots, n\} \ \forall s_{-i} \in S_{-i} \ \forall s_i, s_i' \in S_i$$
$$p_i(s_i, s_{-i}) - p_i(s_i', s_{-i}) > 0 \text{ iff } P(s_i, s_{-i}) - P(s_i', s_{-i}) > 0.$$

Example

- Modified Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,2$

- This game has no potential.
- Ordinal potential

	C	D
C	0	1
D	1	2

Finite Improvement Property (FIP)

Fix a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.
$S := S_1 \times \cdots \times S_n$.

- s_i' is a better response given s if $p_i(s_i', s_{-i}) > p_i(s_i, s_{-i})$.
- A path in S is a sequence $(s^1, s^2, \ldots)$ of joint strategies such that

$$\forall k > 1 \, \exists i \, \exists s_i' \neq s_i^k \; s^{k+1} = (s_i', s_{-i}^k).$$

- A path is an improvement path if it is maximal and for all $k > 1$, $p_i(s^{k+1}) > p_i(s^k)$, where i deviated from s^k.
- G has the finite improvement property (FIP), if every improvement path is finite.
- Note If G has the FIP, then it has a Nash equilibrium.

Ordinal Potentials vs FIP

Example

	L	R
T	$1,0$	$2,0$
B	$2,0$	$0,1$

- This game has the FIP.
- It does not have an ordinal potential.

Proof. Every ordinal potential has to satisfy

$$P(T,L) < P(B,L) < P(B,R) < P(T,R).$$

But $p_2(T,L) = p_2(T,R)$, so $P(T,L) = P(T,R)$.

Generalized Ordinal Potentials

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- Function $P : S_1 \times \cdots \times S_n \to \mathbb{R}$ is a generalized ordinal potential for G if

$$\forall i \in \{1, \ldots, n\} \; \forall s_{-i} \in S_{-i} \; \forall s_i, s_i' \in S_i$$

$$p_i(s_i, s_{-i}) - p_i(s_i', s_{-i}) > 0 \text{ implies } P(s_i, s_{-i}) - P(s_i', s_{-i}) > 0.$$

- Example

	L	R
T	$1, 0$	$2, 0$
B	$2, 0$	$0, 1$

- Generalized Ordinal Potential

	C	D
C	0	3
D	1	2

Generalized Ordinal Potentials vs FIP

Theorem (Monderer and Shapley '96)
Every finite game has a generalized ordinal potential iff it has the FIP.

Proof. $(\Rightarrow)$
The generalized ordinal potential increases along every improvement path.

$(\Leftarrow)$ (Sketch).
An improvement sequence: a prefix of an improvement path.
Assign to each joint strategy s the number of improvement sequences that terminate in it.
Because the game has the FIP this number is finite.
This defines a generalized ordinal potential.

Payoff Functions vs Cost Functions

- Until now we associated with each player a payoff function p_i.
- An alternative: associate with each player a cost function c_i.
- Objective: minimize the cost.
- Translation:

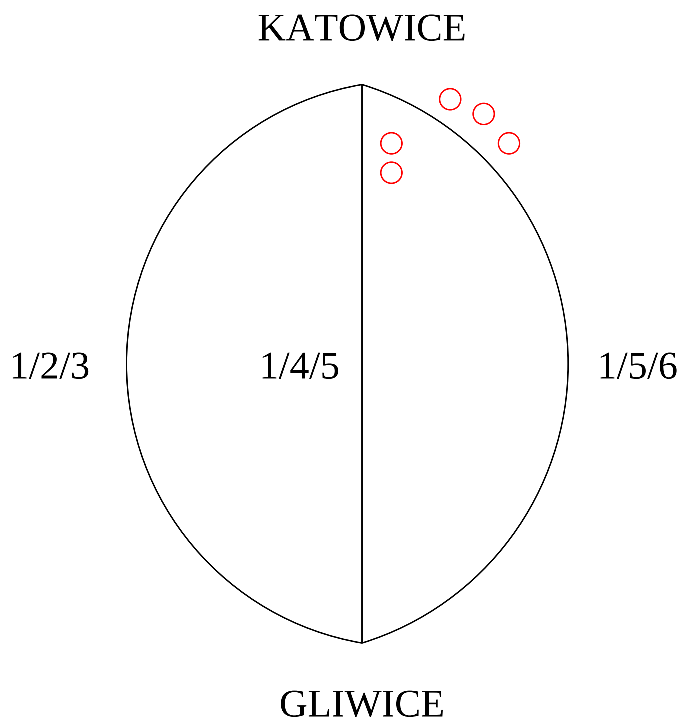
$$p_i(s) := -c_i(s).$$

Congestion Games

- $n > 1$ players,
- Finite set E of facilities (road segments, primary production factors, $\ldots$),
- each strategy is a non-empty subset of E,
- each player has a possibly different set of strategies,
- we use here cost functions c_i instead of payoff functions p_i,
- $d_j : \{1, \ldots, n\} \to \mathbb{R}$ is the delay function for using $j \in E$,
- $d_j(k)$ is the delay for using j when there are k users of j,
- $x_j(s) := |\{r \in \{1, \ldots, n\} \mid j \in s_r\}|$ is the number of users of facility j given s,
- $c_i(s) := \sum_{j \in s_i} d_j(x_j(s))$.

Example

- 5 drivers.
- Each driver chooses a road from Katowice to Gliwice,
- More drivers choose the same road: more delays.
  (1/4/5 $\equiv d(1) = 1, d(2) = 4, d(3) = 5$, etc.)

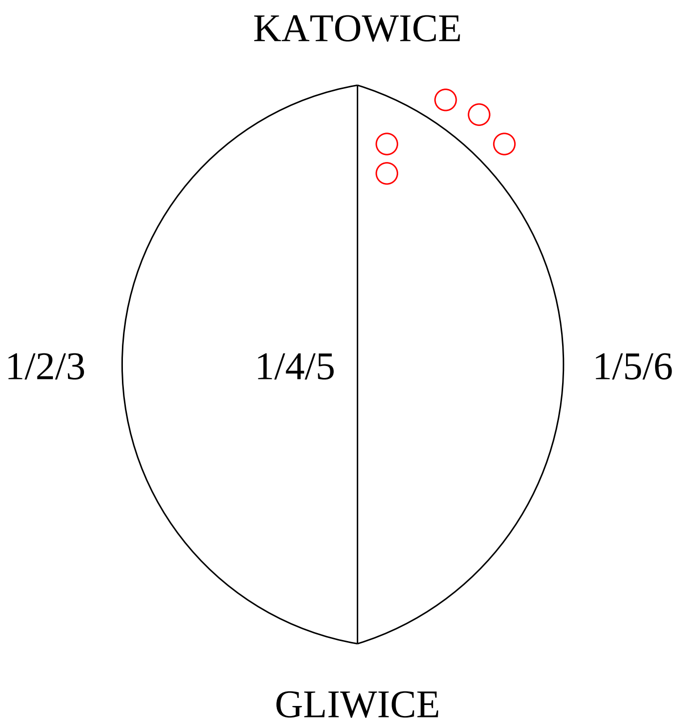


Example as a Congestion Game

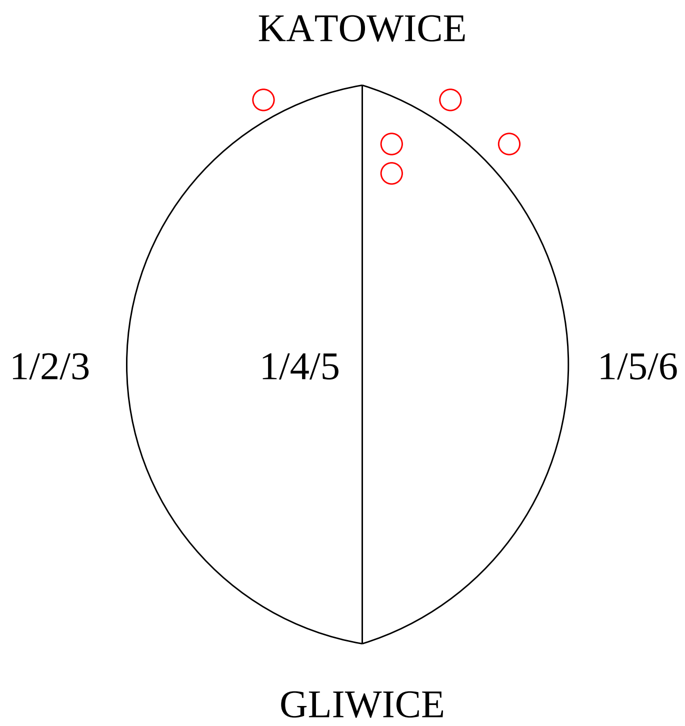
- 5 players,
- 3 facilities (roads),
- each strategy: (a singleton set consisting of) a road,
- cost function:

$$c_i(s) := \begin{cases} 1 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| = 1 \\ 2 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| = 2 \\ 3 & \text{if } s_i = 1 \text{ and } |\{j \mid s_j = 1\}| \geq 3 \\ 1 & \text{if } s_i = 2 \text{ and } |\{j \mid s_j = 2\}| = 1 \\ \dots & \\ 6 & \text{if } s_i = 3 \text{ and } |\{j \mid s_j = 3\}\}| \geq 3 \end{cases}$$

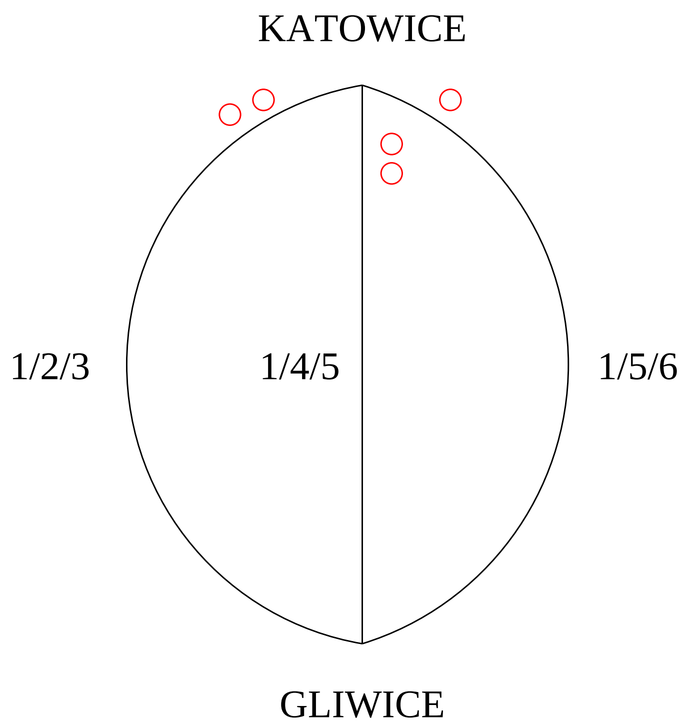
Possible Evolution (1)



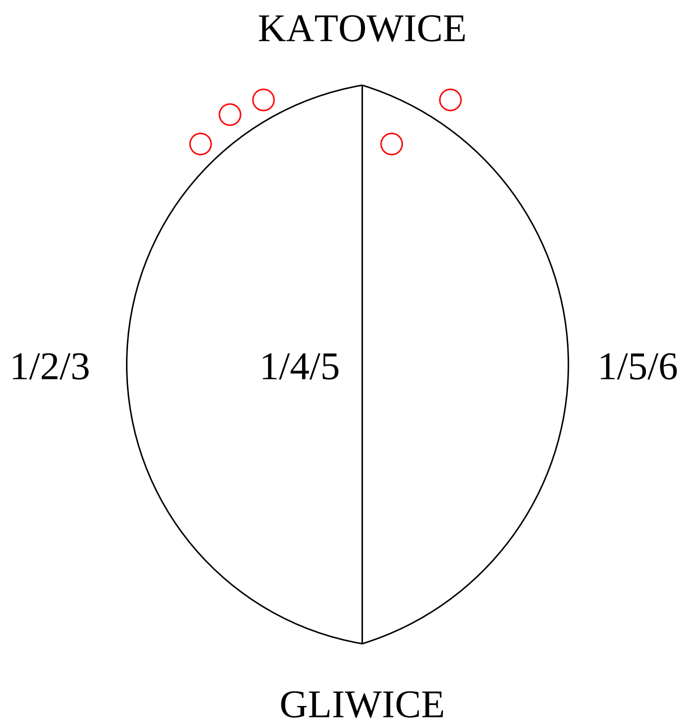
Possible Evolution (2)



Possible Evolution (3)



Possible Evolution (4)

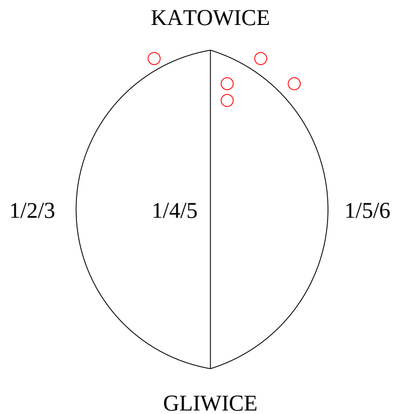


We reached a Nash equilibrium using the best response dynamics.

Congestion Games, ctd

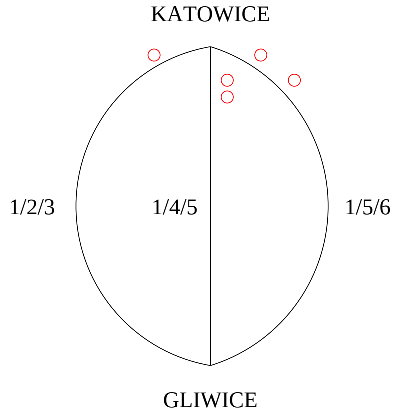
Theorem (Rosenthal, '73)
Every congestion game is a potential game.

Proof for the example game.
Define $P(s)$ to be the sum of the accumulated delays on all roads.

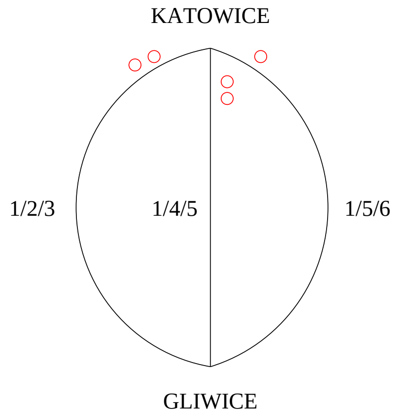


Here $P(s) :=$
$1(\text{for left road}) + 1 + 4(\text{for middle road}) + 1 + 5(\text{for right road}) = 12$.

Congestion Games, ctd



Here $P(s) = 12$.



Here $P(s) = 12 - 5 + 2 = 9$.

So both the switching player's cost function and the potential decreased by 3.

Congestion Games, ctd

General argument.

$$P(s) := \sum_{j \in s_1 \cup \ldots \cup s_n} \sum_{k=1}^{x_j(s)} d_j(k),$$

where (recall) $x_j(s) = |\{r \in \{1, \ldots, n\} \mid j \in s_r\}|$,

is a potential function.
(For the proof see the last page of the pages from S. Tijs, Introduction to Game Theory on the home page of the course `http://homepages.cwi.nl/~apt/stra13/`.)

Conclusion Every congestion game has a Nash equilibrium.

Fair Cost Sharing Games

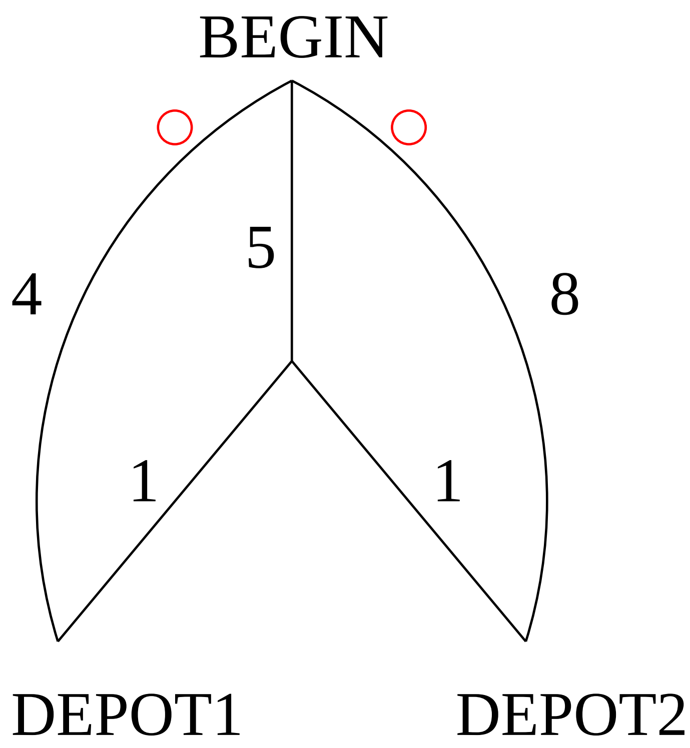
A special case of congestion games.

- $c_j \in \mathbb{R}$ is the cost of facility $j \in E$.
- Recall: $x_j(s)$ is the number of players using facility j in s.
- Use $d_j(x_j(s)) := \frac{c_j}{x_j(s)}$ in the definition of the congestion game.
- So the cost of facility $j \in E$ is evenly shared. Consequently

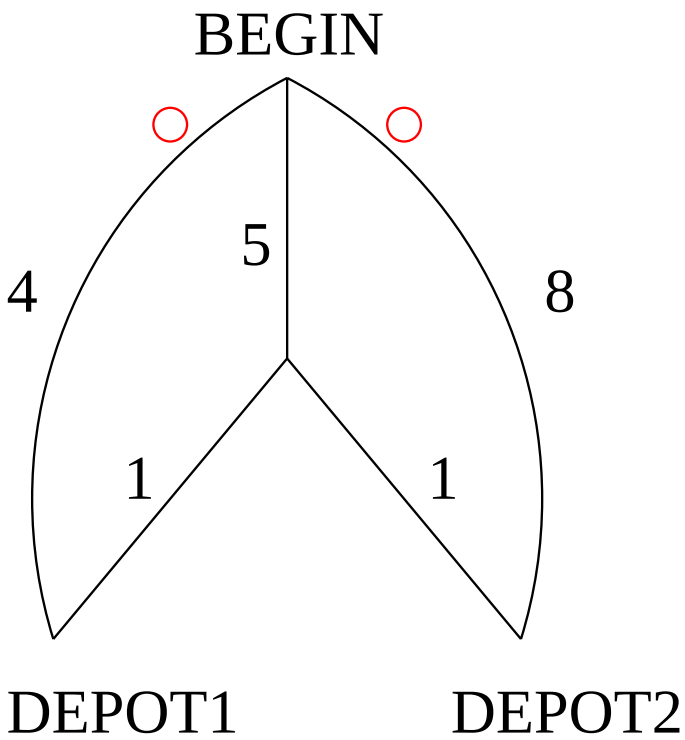
$$c_i(s) := \sum_{j \in s_i} \frac{c_j}{x_j(s)}.$$

Example

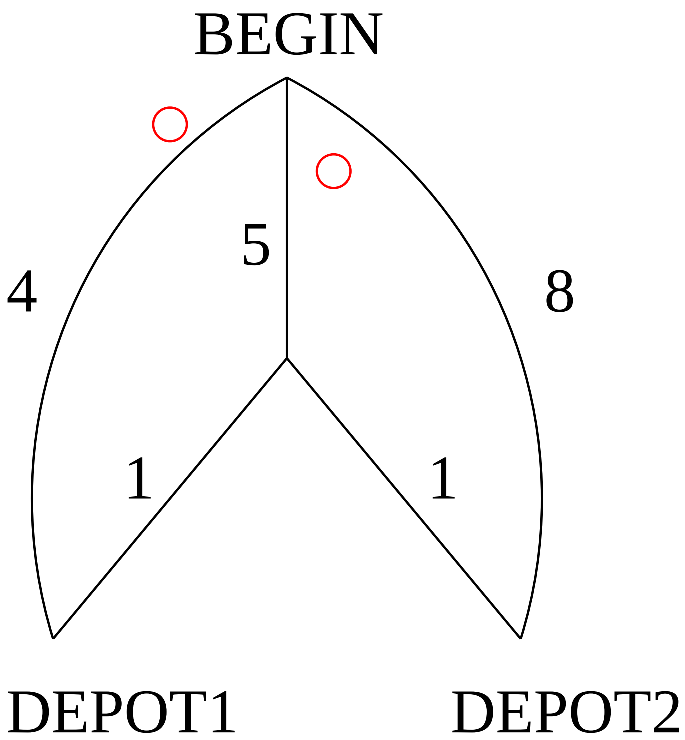
- 2 drivers.
- Each driver chooses a route from BEGIN to his own depot.
- Fair congestion game, so the costs are equally divided.



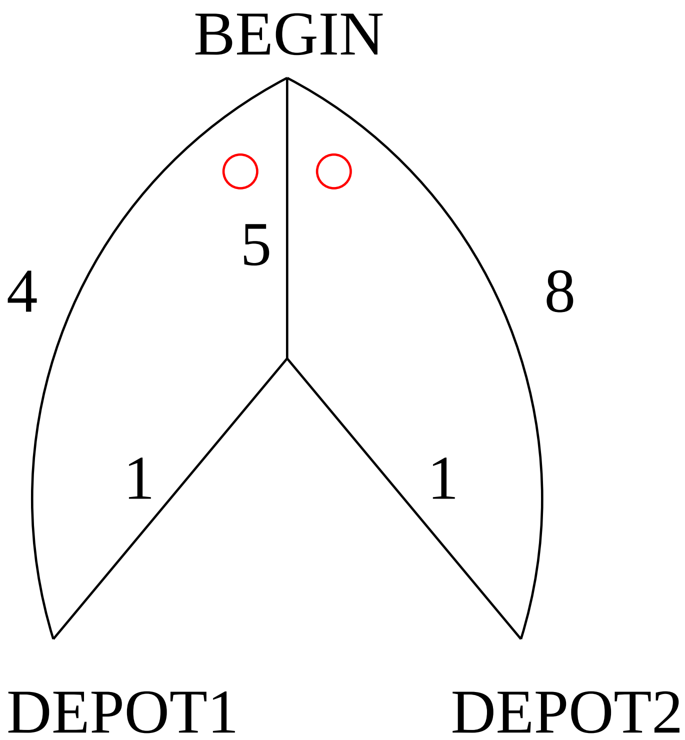
Possible Evolution (1)



Possible Evolution (2)



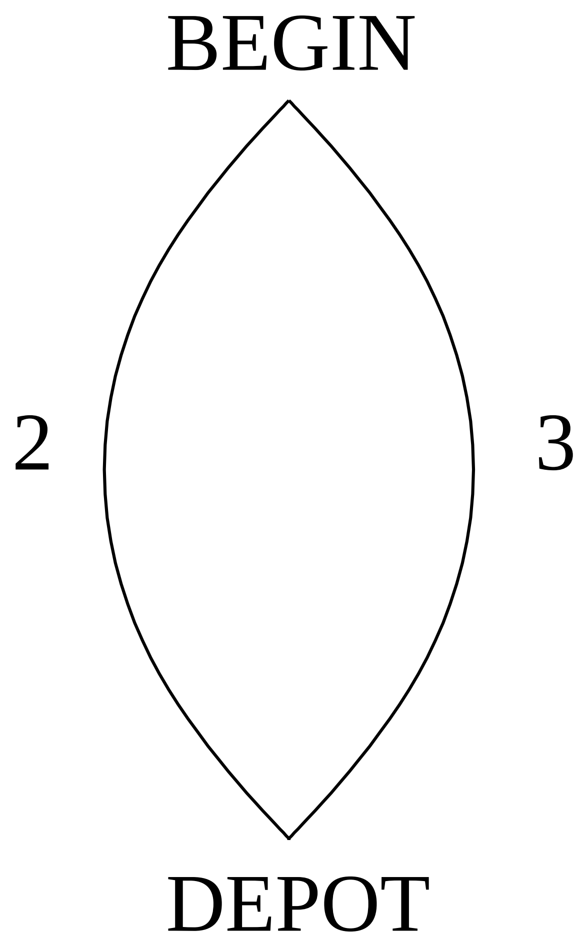
Possible Evolution (3)



A Nash equilibrium is reached.
It is a unique Nash equilibrium and also a social optimum.

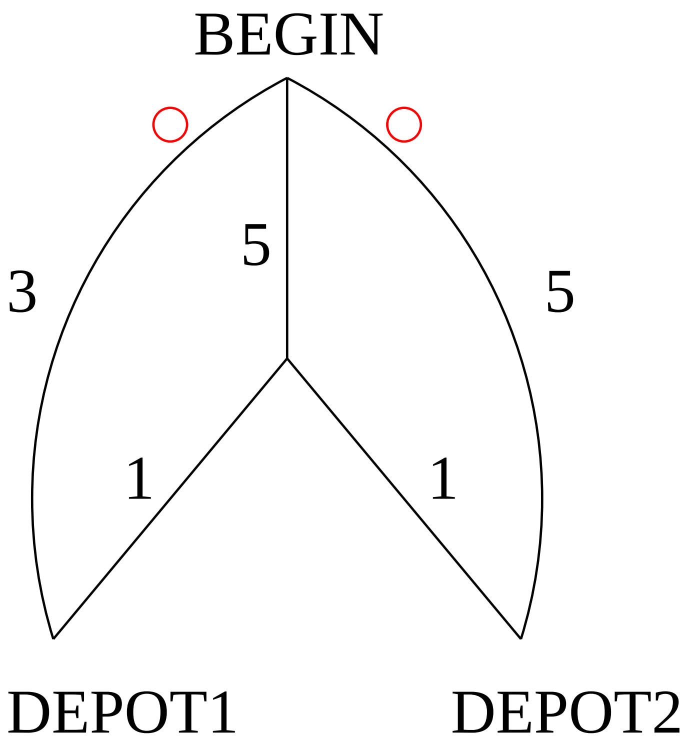
Multiple Nash Equilibria

Two players.



Two Nash equilibria.

Another Example

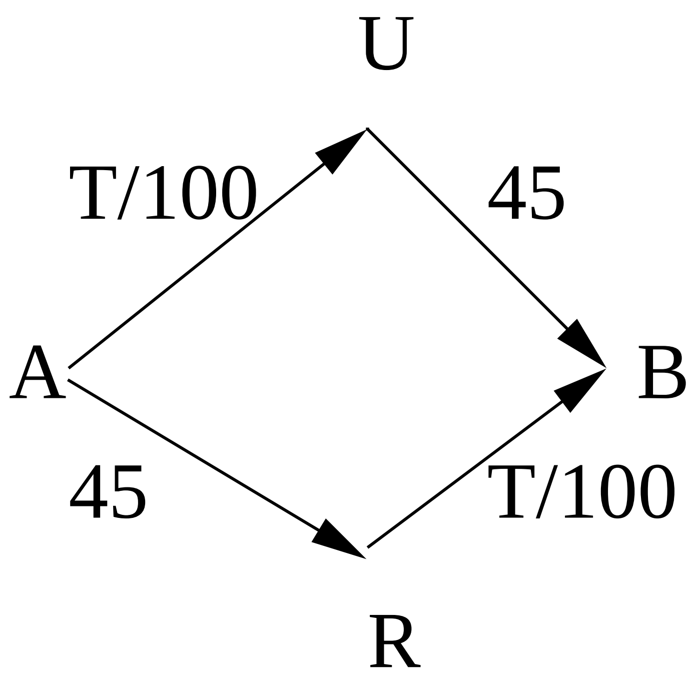


- Unique Nash equilibrium, with the social cost 8.
- Cost of the social optimum: 7.

Another Example: Congestion Game

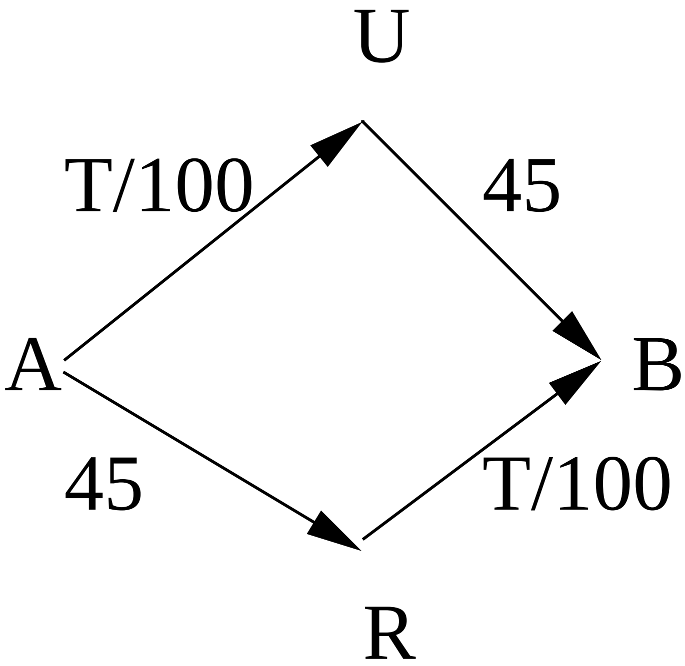
Assumptions:

- 4000 drivers drive from A to B.
- Each driver has 2 options (strategies).



Problem: Find a Nash equilibrium (T = number of drivers).

Nash Equilibrium

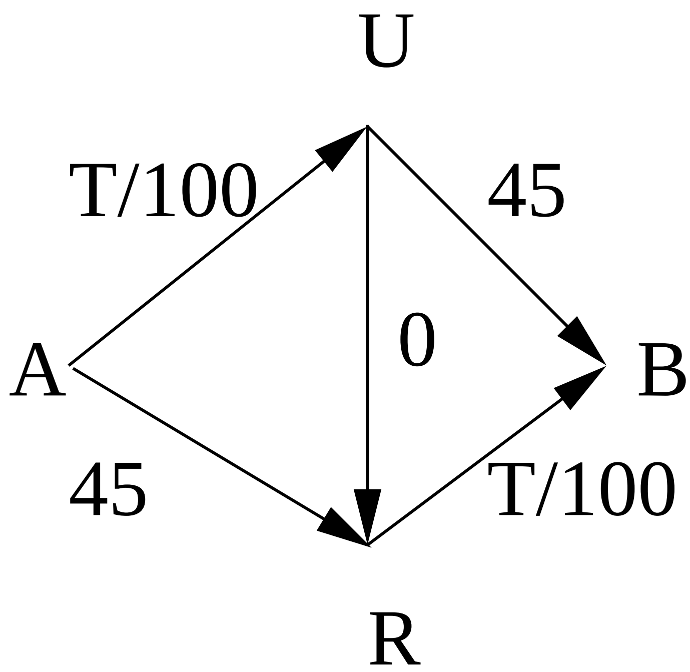


Answer: 2000/2000.

Travel time: 2000/100 + 45 = 45 + 2000/100 = 65.

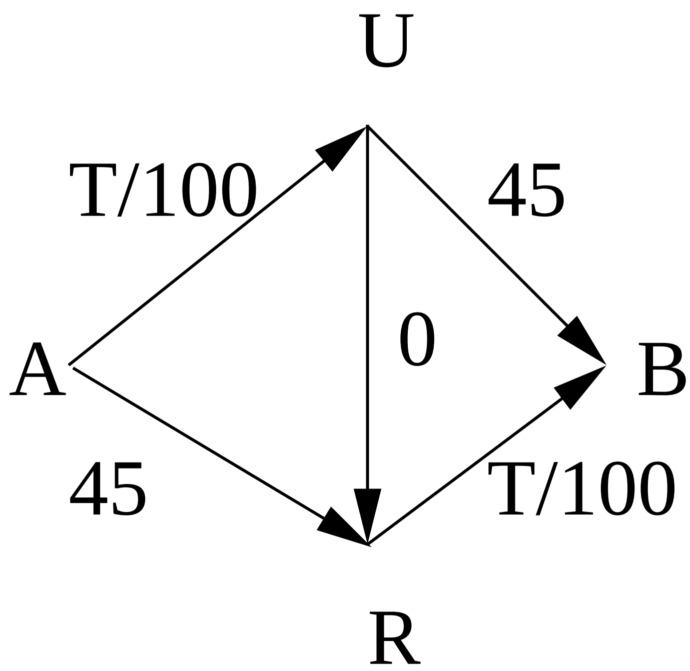
Braess Paradox

- Add a fast road from U to R.
- Each driver has now 3 options (strategies):
  A - U - B,
  A - R - B,
  A - U - R - B.



Problem: Find a Nash equilibrium.

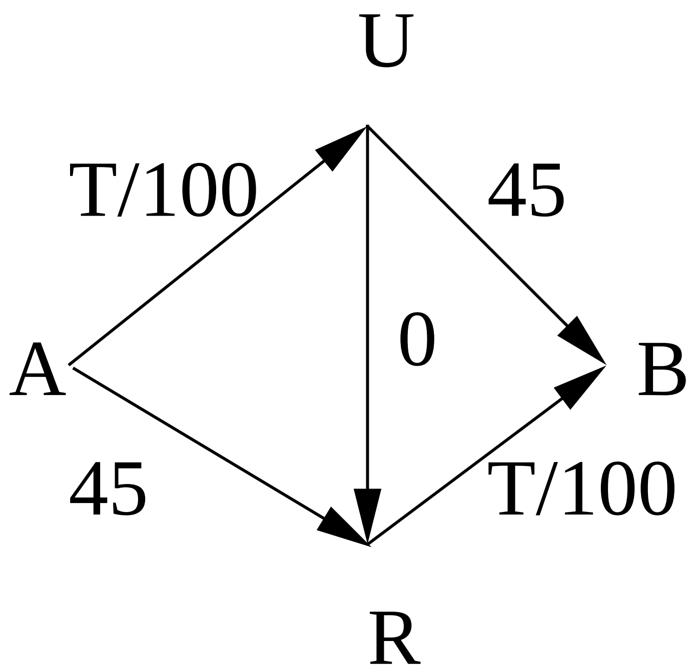
Nash Equilibrium



Answer: Every driver will choose the road A - U - R - B.

Why?: The road A - U - R - B is a strictly dominant strategy. So every best response dynamics terminates after ≤ 4000 steps and has a unique outcome.

Small Complication



- Travel time: 4000/100 + 4000/100 = 80!
- Braess paradox: Adding a new road results in strictly longer travel times.
- Formally: adding a new strategy resulted in a game with a unique Nash equilibrium that is strictly worse for everybody that the original unique Nash equilibrium.

Does it happen?

from Wikipedia ('Braess Paradox'):

- In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
- In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.
- In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.
- In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.

Price of Stability

Definition Price of Stability (PoS):

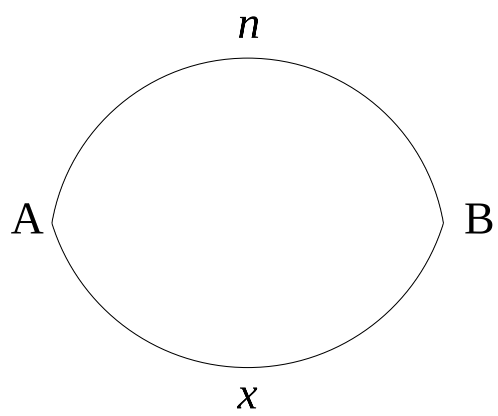
$$\frac{\text{social cost of the best Nash equilibrium}}{\text{social cost of the social optimum}}$$

Here:

best Nash equilibrium: one with the minimum social cost.
Social optimum: joint strategy with the minimum social cost.

Question: What is the price of stability for the congestion games and for the fair cost sharing games?

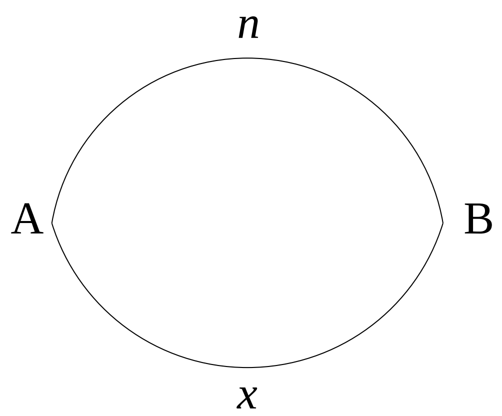
Example



n - (even) number of players.
x - number of drivers on the bottom road.

- Two Nash equilibria
  $1/(n-1)$, with the social cost $n + (n-1)^2$.
  $0/\,n$, with the social cost n^2.
- Social optimum
  Take $f(x) = x \cdot x + (n - x) \cdot n = x^2 - n \cdot x + n^2$.
  We want to find a minimum of f.
  $f'(x) = 2x - n$, so $f'(x) = 0$ if $x = \frac{n}{2}$.

Example



- Best Nash equilibrium
  $1/(n-1)$, with social cost $n + (n-1)^2$.
- Social optimum
  $f(x) = x^2 - n \cdot x + n^2$.
  Social optimum = $f(\frac{n}{2}) = \frac{3}{4}n^2$.
- PoS = $(n + (n-1)^2)/\frac{3}{4}n^2 = \frac{4}{3}\frac{n+(n-1)^2}{n^2}$.
- $\lim_{n\to\infty}$ PoS = $\frac{4}{3}$.

Price of Stability I

- Theorem (Roughgarden and Tárdos, 2002)
  Suppose delay functions (e.g., $T/100$) are linear.
  Then the price of stability for the congestion games is $\leq \frac{4}{3}$.

Harmonic Numbers

- $H(n) = \sum_{i=1}^{n} \frac{1}{n}$.
- Theorem (Oresme, around 1350) $\lim_{n\to\infty} H(n) = \infty$.
- Theorem (Euler, 1734) For some constant γ

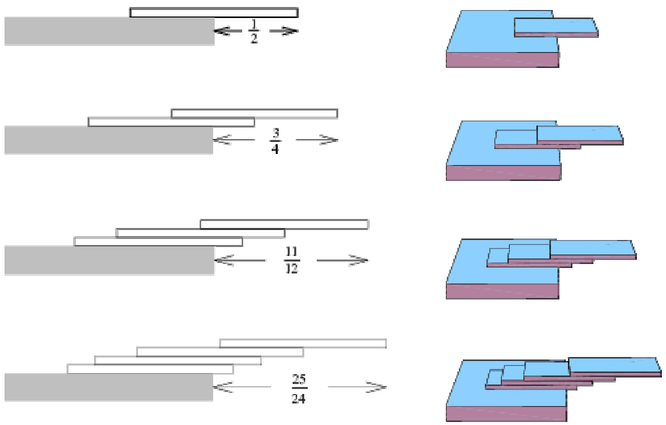
$$\lim_{n\to\infty} (H(n) - \ln(n)) = \gamma.$$

Proof (Nicolas Oresme)

$$
\begin{aligned}
& 1+1/2+1/3+1/4+1/5+1/6+1/7+1/8+\dots \\
= \; & 1+1/2+(1/3+1/4)+(1/5+1/6+1/7+1/8)+\dots \\
> \; & 1+1/2+(1/4+1/4)+(1/8+1/8+1/8+1/8)+\dots \\
= \; & 1+1/2+\qquad 1/2 \qquad + \qquad\qquad 1/2 \qquad\qquad +\dots
\end{aligned}
$$

Harmonic Numbers: An Application

Problem: Build the longest ‘trampoline’ from the books:



Question: How many books one needs to double the length?

Answer: smallest n such that $\frac{1}{2}H(n) \geq 2$.

$\frac{1}{2}H(30) = 1.99749$, $\frac{1}{2}H(31) = 2.01362$.

So the answer is 31.

Price of Stability II

- Theorem (Anshelevich et al, 2004)
  The price of stability for the fair cost sharing games is $\leq H(n)$.