# Selfishness Level of Strategic Games

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# Altruistic Games

- Given $G := (N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N})$ and $\alpha \geq 0$.
- $G(\alpha) := (N, \{S_i\}_{i\in N}, \{r_i\}_{i\in N})$, where

$$r_i(s) := p_i(s) + \alpha SW(s).$$

- When $\alpha > 0$ the payoff of each player in $G(\alpha)$ depends on the social welfare of the players.
- $G(\alpha)$ is an altruistic version of $G$.

# Selfishness Level (1)

- $G$ is $\alpha$-selfish if a Nash equilibrium of $G(\alpha)$ is a social optimum of $G(\alpha)$.
- Selfishness level of $G$:

$$\inf\{\alpha \in \mathbb{R}_+ \mid G \text{ is } \alpha\text{-selfish}\}.$$

Recall $\inf(\emptyset) = \infty$.

- Selfishness level of $G$ is $\alpha^+$ iff the selfishness level of $G$ is $\alpha \in \mathbb{R}_+$ but $G$ is *not* $\alpha$-selfish.

# Selfishness Level (2)

Intuition

Selfishness level quantifies the minimal share of social welfare needed to induce the players to choose a social optimum.

# Three Examples (1)

## Prisoner's Dilemma

| | $C$ | $D$ |
|---|---|---|
| $C$ | $2,2$ | $0,3$ |
| $D$ | $3,0$ | $1,1$ |

## The Battle of the Sexes

| | $F$ | $B$ |
|---|---|---|
| $F$ | $2,1$ | $0,0$ |
| $B$ | $0,0$ | $1,2$ |

## Matching Pennies

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1,-1$ | $-1,\ \ 1$ |
| $T$ | $-1,\ \ 1$ | $1,-1$ |

# Three Examples (2)

**Prisoner's Dilemma:** selfishness level is 1.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $2,2$ | $0,3$ |
| $D$ | $3,0$ | $1,1$ |

| | $C$ | $D$ |
|---|---|---|
| $C$ | $6,6$ | $3,6$ |
| $D$ | $6,3$ | $3,3$ |

**The Battle of the Sexes:** selfishness level is 0.

| | $F$ | $B$ |
|---|---|---|
| $F$ | $2,1$ | $0,0$ |
| $B$ | $0,0$ | $1,2$ |

**Matching Pennies:** selfishness level is $\infty$.

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1,-1$ | $-1,\ \ 1$ |
| $T$ | $-1,\ \ 1$ | $1,-1$ |

# Another Example

## Game with a bad Nash equilibrium

| | $H$ | $T$ | $E$ |
|---|---|---|---|
| $H$ | $1,-1$ | $-1,\ \ 1$ | $-1,-1$ |
| $T$ | $-1,\ \ 1$ | $1,-1$ | $-1,-1$ |
| $E$ | $-1,-1$ | $-1,-1$ | $-1,-1$ |

- The unique Nash equilibrium is $(E,E)$.
- The selfishness level of this game is $\infty$.

# Yet Another Example

## Game with no Nash equilibrium

Consider $G$ on the left and $G(1)$ on the right.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $2,2$ | $2,0$ |
| $D$ | $3,0$ | $1,1$ |

| | $C$ | $D$ |
|---|---|---|
| $C$ | $6,6$ | $4,2$ |
| $D$ | $6,3$ | $3,3$ |

- $G$ has no Nash equilibrium, while in $G(1)$ the social optimum, $(C,C)$, is also a Nash equilibrium.
- For $\alpha < 1$, $(C,C)$ is also a social optimum of $G(\alpha)$ but not a Nash equilibrium.
- So the selfishness level of the game $G$ is 1.

# Invariance of Selfishness Level

Lemma Consider a game $G$ and $\alpha \geq 0$.

- For every $a$, $G$ is $\alpha$-selfish iff $G + a$ is $\alpha$-selfish,
- For every $a > 0$, $G$ is $\alpha$-selfish iff $aG$ is $\alpha$-selfish.

Conclusion Selfishness level is invariant under positive linear transformations of the payoff functions.

# Selfishness Level vs Price of Stability (1)

- Recall
  Price of stability = $SW(s)/SW(s')$,
  where $s$ is a social optimum and $s'$ a Nash equilibrium with the highest social welfare.
- Note
  Selfishness level of a finite game is 0 iff price of stability is 1.

# Selfishness Level vs Price of Stability (2)

Theorem For every finite $\alpha > 0$ and $\beta > 1$ there is a finite game with selfishness level $\alpha$ and price of stability $\beta$.

Proof Consider $G$:

| | $C$ | $D$ |
|---|---|---|
| $C$ | $1, 1$ | $0, \frac{2\alpha+1}{\alpha+1}$ |
| $D$ | $\frac{2\alpha+1}{\alpha+1}, 0$ | $\frac{1}{\beta}, \frac{1}{\beta}$ |

In each $G(\gamma)$ with $\gamma \geq 0$, $(C,C)$ is the unique social optimum. Consider $G(\gamma)$ and stipulate that $(C,C)$ is its Nash equilibrium. This leads to

$$1 + 2\gamma \geq (\gamma + 1)\frac{2\alpha + 1}{\alpha + 1}.$$

This is equivalent to $\gamma \geq \alpha$. So the selfishness level is $\alpha$. The price of stability is $\beta$.

# Selfishness Level can be $\alpha^+$

Theorem There exists a game that is $0^+$-selfish (so $\alpha$-selfish for every $\alpha > 0$, but is not 0-selfish).

Proof idea
Plug the above games for each $\alpha > 0$ and fixed $\beta > 1$ in:

| | | | | | |
|---|---|---|---|---|---|
| ... | ... | 0 | 0 | 0 | 0 |
| ... | ... | 0 | 0 | 0 | 0 |
| 0 | 0 | ... | ... | 0 | 0 |
| 0 | 0 | ... | ... | 0 | 0 |
| 0 | 0 | 0 | 0 | ... | ... |
| 0 | 0 | 0 | 0 | ... | ... |

# Alternative Definitions (1)

**A:** For every $\alpha \geq 0$, $G(\alpha) := (N, \{S_i\}_{i \in N}, \{r_i^\alpha\}_{i \in N})$ with

$$r_i^\alpha(s) = p_i(s) + \alpha SW(s) \quad \forall i \in N.$$

**B:** For every $\beta \in [0,1]$, $G(\beta) := (N, \{S_i\}_{i \in N}, \{r_i^\beta\}_{i \in N})$ with

$$r_i^\beta(s) = (1-\beta) p_i(s) + \frac{\beta}{n} SW(s) \quad \forall i \in N.$$

**C:** For every $\gamma \in [0,1]$, $G(\gamma) := (N, \{S_i\}_{i \in N}, \{r_i^\gamma\}_{i \in N})$ with

$$r_i^\gamma(s) = (1-\gamma) p_i(s) + \gamma SW(s) \quad \forall i \in N.$$

**D:** For every $\delta \in [0,1]$, $G(\delta) := (N, \{S_i\}_{i \in N}, \{r_i^\delta\}_{i \in N})$ with

$$r_i^\gamma(s) = (1-\delta) p_i(s) + \delta (SW(s) - p_i(s)) \quad \forall i \in N.$$

# Alternative Definitions (2)

Theorem

Consider $G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ and its altruistic versions defined according to models A, B, C and D.

(i) $G$ is $\alpha$-selfish with $\alpha \in \mathbb{R}_+$ iff $G$ is $\beta$-selfish with $\beta = \frac{\alpha n}{1+\alpha n} \in [0,1]$.

(ii) $G$ is $\alpha$-selfish with $\alpha \in \mathbb{R}_+$ iff $G$ is $\gamma$-selfish with $\gamma = \frac{\alpha}{1+\alpha} \in [0,1]$.

(iii) $G$ is $\alpha$-selfish with $\alpha \in \mathbb{R}_+$ iff $G$ is $\delta$-selfish with $\delta = \frac{\alpha}{1+2\alpha} \in [0, \frac{1}{2}]$.

# Stable Social Optima

- Social optimum $s$ stable if no player is better off by unilaterally deviating to another social optimum.
- That is, $s$ is stable if for all $i \in N$ and $s'_i \in S_i$

  if $(s'_i, s_{-i})$ is a social optimum, then $p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$.
- Notes
  - If $s$ is a unique social optimum, then it is stable.
  - Stable social optima don't need to exist: take the Matching Penny game.

# Characterization Result

Player $i$'s appeal factor of $s'_i$ given the social optimum $s$:

$$\mathit{AF}_i(s'_i, s) := \frac{p_i(s'_i, s_{-i}) - p_i(s_i, s_{-i})}{SW(s_i, s_{-i}) - SW(s'_i, s_{-i})}.$$

Theorem

- The selfishness level of $G$ is finite iff a stable social optimum $s$ exists for which
  $\alpha(s) := \max_{i \in N, s'_i \in U_i(s)} \mathit{AF}_i(s'_i, s)$
  is finite, where
  $U_i(s) := \{s'_i \in S_i \mid p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})\}$.
- If the selfishness level of $G$ is finite, then it equals $\min_{s \in SSO} \alpha(s)$, where SSO is the set of stable social optima.

# Some Observations

- If $G$ is finite, then its selfishness level is finite iff it has a stable social optimum.
- Selfishness level can be unbounded.

  Theorem For each $f : \mathbb{N} \to \mathbb{R}_+$ there exists a class of games $G_n$ for $n$ players, such that the selfishness level of $G_n$ is $f(n)$.

# Some Examples

**Prisoner's dilemma for $n$ players** Each $S_i = \{0, 1\}$,

$$p_i(s) := 1 - s_i + 2 \sum_{j \neq i} s_j.$$

Proposition Selfishness level is $\frac{1}{2n-3}$.

**Traveler's dilemma** Two players, $S_i = \{2, \ldots, 100\}$,

$$p_i(s) := \begin{cases} s_i & \text{if } s_i = s_{-i} \\ s_i + 2 & \text{if } s_i < s_{-i} \\ s_{-i} - 2 & \text{otherwise.} \end{cases}$$

Proposition Selfishness level is $\frac{1}{2}$.

# Public Goods Game

- $n$ players,
- $b \in \mathbb{R}_+$: fixed budget,
- $c > 1$: a multiplier,
- $S_i = [0, b]$,
- $p_i(s) := b - s_i + \frac{c}{n} \sum_{j \in N} s_j$.

Proposition Selfishness level is $\max\left\{0, \frac{1-\frac{c}{n}}{c-1}\right\}$.

Notes

- Free riding: contributing 0 (it is a dominant strategy).
- For fixed $c$ temptation to free ride increases with $n$.
- For fixed $n$ temptation to free ride decreases as $c$ increases.

# Potential Games

$$G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$$

is a generalized ordinal potential game if for some
$P : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$
for all $i \in N$, $s_{-i} \in S_{-i}$ and $s_i, s'_i \in S_i$

$$p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}) \text{ implies } P(s_i, s_{-i}) > P(s'_i, s_{-i}).$$

Theorem Every finite generalized ordinal potential game has a finite selfishness level.

Proof Each social optimum with the largest potential is a stable social optimum.

# Fair Cost Sharing Games (1)

Fair cost sharing game: $G = (N, E, \{S_i\}_{i \in N}, \{c_e\}_{e \in E})$, where

- $E$ is the set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player $i$, i.e., each $s_i \subseteq E$,
- $c_e \in \mathbb{Q}_+$ is the cost of facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility $e$ in $s$.
- The cost of facility $e \in E$ is evenly shared. So $c_i(s) := \sum_{e \in s_i} \frac{c_e}{x_e(s)}$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

# Fair Cost Sharing Games (2)

Singleton cost sharing game: for each $s_i$, $|s_i| = 1$.

- $c_{\max} := \max_{e \in E} c_e$,
- $c_{\min} := \min_{e \in E} c_e$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$
(maximum number of facilities a player can choose).

Proposition Selfishness level of

- a singleton cost sharing game is $\leq \max\{0, \frac{1}{2}\frac{c_{\max}}{c_{\min}} - 1\}$. The bound is tight.
- a fair cost sharing game with non-negative integer costs is $\leq \max\{0, \frac{1}{2}Lc_{\max} - 1\}$. The bound is tight.

# Congestion Games

Congestion game: $G = (N, E, \{S_i\}_{i \in N}, \{d_e\}_{e \in E})$, where

- $E$ is a finite set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player $i$,
- $d_e \in \mathbb{N}$ is the delay function for facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility $e$ in $s$.
- The goal of a player is to minimize his individual cost $c_i(s) := \sum_{e \in s_i} d_e(x_e(s))$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

# Linear Congestion Games

Linear congestion game: each delay function is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{R}_+$.

- $\Delta_{\max} := \max_{e \in E}(a_e + b_e)$,
- $\Delta_{\min} := \min_{e \in E}(a_e + b_e)$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$
  (maximum number of facilities a player can choose).

Proposition Selfishness level of

- a linear congestion game with non-negative integer coefficients is $\leq \max\{0, \frac{1}{2}(L\Delta_{\max} - \Delta_{\min} - 1)\}$. This bound is tight.

# Cournot Competition (1)

- One infinitely divisible product (oil),
- $n$ companies decide simultaneously how much to produce,
- price is decreasing in total output.

Each $S_i = \mathbb{R}_+$,

$$p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$$

for some $a, b, c$, where $a > c$ and $b > 0$.

The price of the product: $a - b\sum_{j=1}^{n} s_j$.

The production cost: $cs_i$.

# Cournot Competition (2)

- $p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$
- Unique Nash equilibrium:
  $s$, with each $s_i = \frac{a-c}{b(n+1)}$.
  $SW(s) = \frac{(a-c)^2}{b} \cdot \frac{n}{(n+1)^2}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$.
  $SW(s) = \frac{(a-c)^2}{4b}$.
- Note Price of stability converges to $\infty$.

Proposition For each $n > 1$ the selfishness level is $\infty$.

# Tragedy of the Commons (1)

- Contiguous common resource (e.g. shared bandwidth),
- Each $S_i = [0,1]$,
- $s_i$: chosen fraction of the common resource
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Intuition: the payoff degrades when the resource is overused.

# Tragedy of the Commons (2)

- $$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
- Best Nash equilibrium:
  $s$, with each $s_i = \frac{1}{n+1}$.
  $SW(s) = \frac{n}{(n+1)^2}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{1}{2}$.
  $SW(s) = \frac{1}{4}$.
- Note Price of stability converges to $\infty$.

Proposition For each $n > 1$ the selfishness level is $\infty$.

# Bertrand Competition

- One product for sale.
- 2 companies simultaneously select their prices.
- The product is sold by the company that chose a lower price.

Each $S_i = [c, \frac{a}{b})$, where $c < \frac{a}{b}$.
(So $s_i - c \geq 0$ and $a - bs_i > 0$ for $s_i \in S_i$.)

$$p_i(s_i, s_{3-i}) := \begin{cases} (s_i - c)(a - bs_i) & \text{if } c < s_i < s_{3-i} \\ \frac{1}{2}(s_i - c)(a - bs_i) & \text{if } c < s_i = s_{3-i} \\ 0 & \text{otherwise.} \end{cases}$$

The demand for the product: $a - bs_i$.

The marginal production cost: $c$.

Proposition The selfishness level is $\infty$.

# Some Quotations

- `The intelligent way to be selfish is to work for the welfare of others.`

  *Microeconomics: Behavior, Institutions, and Evolution*, S. Bowles '04.

- `An excellent way to promote cooperation in a society is to teach people to care about the welfare of others.`

  *The Evolution of Cooperation*, R. Axelrod, '84.

THANK YOU