Social Network Games

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Joint work with Sunil Simon

Social Networks

- Facebook,
- Hyves,
- LinkedIn,
- Nasza Klasa,
- ...

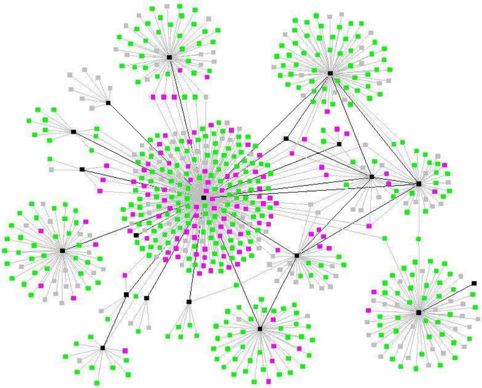
But also . . .

An area with links to

- sociology (spread of patterns of social behaviour)
- economics (effects of advertising, emergence of 'bubbles' in financial markets, . . .),
- epidemiology (epidemics),
- computer science (complexity analysis),
- mathematics (graph theory).

Example 1

(From D. Easley and J. Kleinberg, 2010).



Spread of the tuberculosis outbreak.

Example 2

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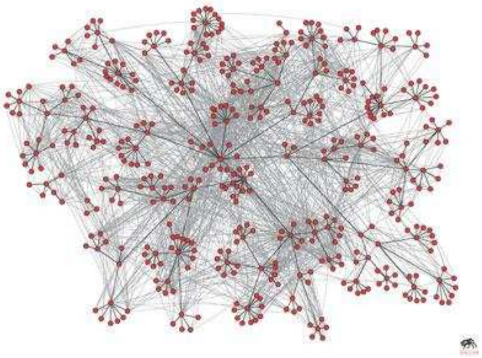
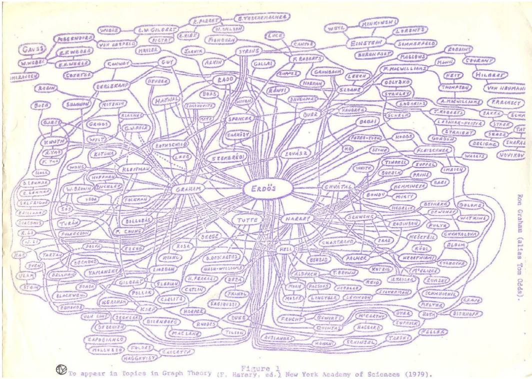


Pattern of e-mail communication among 436 employees of HP Research Lab.

Example 3

(From D. Easley and J. Kleinberg, 2010).



Collaboration of mathematicians centered on Paul Erdős.
Drawing by Ron Graham.

Social networks

Essential components of our model

- Finite set of agents.
- Influence of "friends".
- Finite product set for each agent.
- Resistance level in (threshold for) adopting a product.

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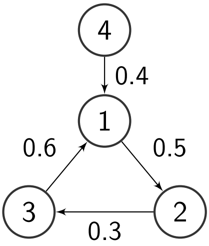

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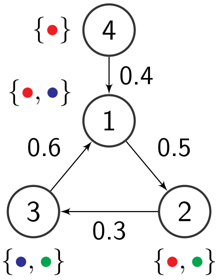

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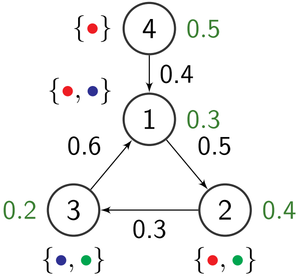

The model

Social network [Apt, Markakis 2011]

- Weighted directed graph: $G = (V, \rightarrow, w)$, where
  V: a finite set of agents,
  $w_{ij} \in (0, 1]$: weight of the edge $i \rightarrow j$.
- Products: A finite set of products $\mathcal{P}$.
- Product assignment: $P : V \rightarrow 2^{\mathcal{P}} \setminus \{\emptyset\}$;
  assigns to each agent a non-empty set of products.
- Threshold function: $\theta(i, t) \in (0, 1]$, for each agent i and product $t \in P(i)$.

- Neighbours of node i: $\{j \in V \mid j \rightarrow i\}$.
- Source nodes: Agents with no neighbours.

The associated strategic game

Interaction between agents: Each agent i can adopt a product from the set $P(i)$ or choose not to adopt any product (t_0).

Social network games

- Players: Agents in the network.
- Strategies: Set of strategies for player i is $P(i) \cup \{t_0\}$.
- Payoff: Fix $c > 0$.
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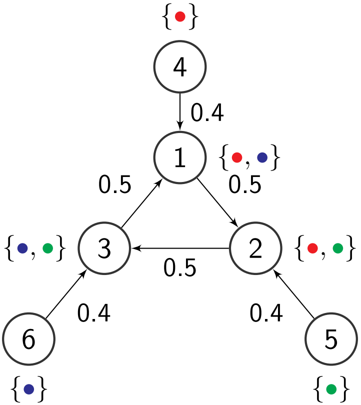
  - if $i \notin source(\mathcal{S})$, $p_i(s) =$
    $\begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i,t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$

  $\mathcal{N}_i^t(s)$: the set of neighbours of i who adopted in s the product t.

Example



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- $\mathcal{P} = \{\bullet, \bullet, \bullet\}$

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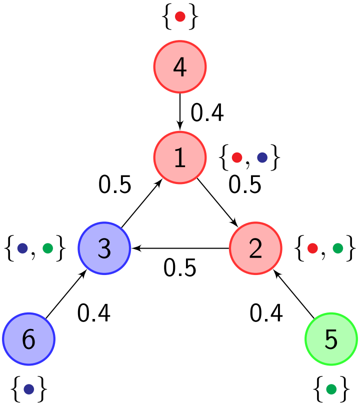

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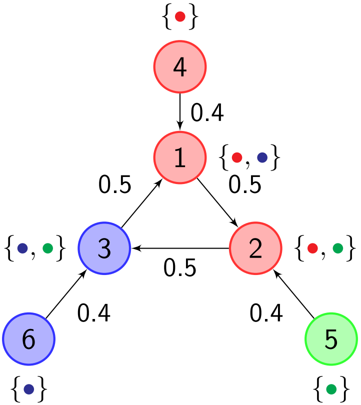

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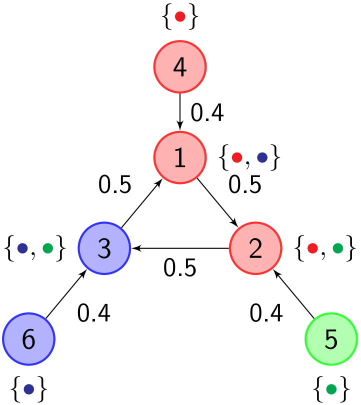

Payoff:

- $p_4(s) = p_5(s) = p_6(s) = c$
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- $p_2(s) = 0.5 - 0.3 = 0.2$
- $p_3(s) = 0.4 - 0.3 = 0.1$

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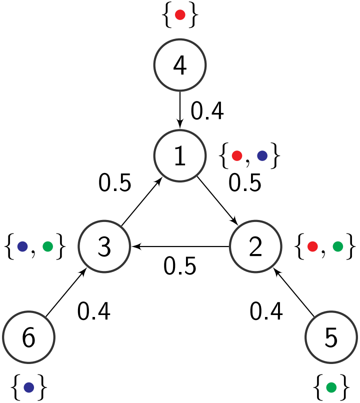
Social network games

Properties

- Graphical game: The payoff for each player depends only on the choices made by his neighbours.
- Join the crowd property: The payoff of each player weakly increases if more players choose the same strategy.

Does Nash equilibrium always exist?



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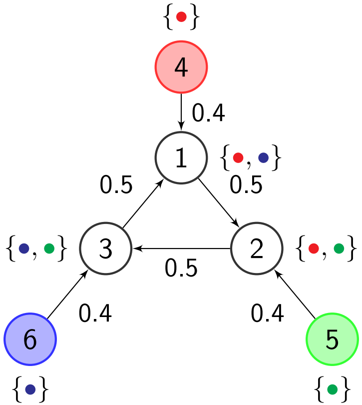

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- Source nodes can ensure a payoff of $c > 0$.
- Each player on the cycle can ensure a payoff of at least 0.1.

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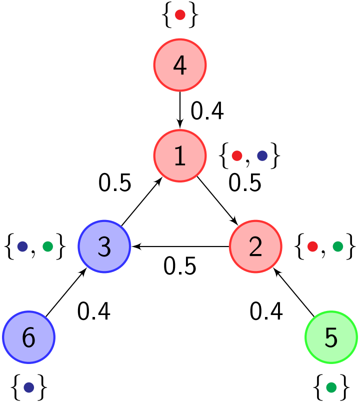

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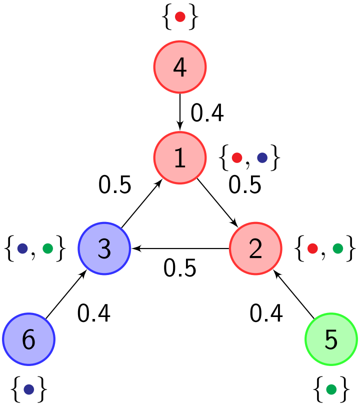

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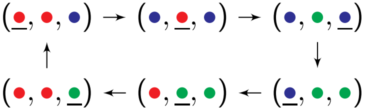
Best response dynamics



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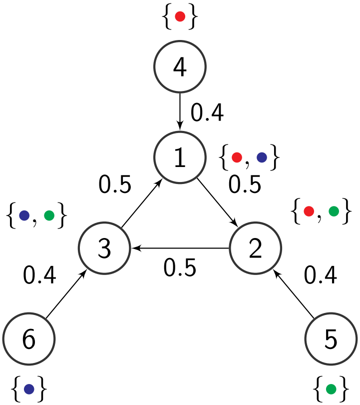
- Source nodes can ensure a payoff of $c > 0$.
- Each player on the cycle can ensure a payoff of at least 0.1.

Reason: Players keep switching between the products.

Nash equilibrium

Recall the network with no Nash equilibrium:

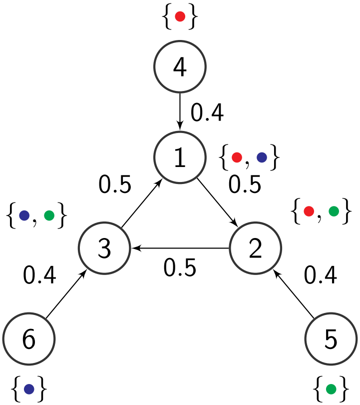


Nash equilibrium (ctd)

Theorem. If there exists $X \subseteq \mathcal{P}$ where $|X| \leq 2$ such that for all source nodes i, $P(i) \cap X \neq \emptyset$ then $\mathcal{S}$ has a Nash equilibrium.

Corollary. If there are at most two products, then a Nash equilibrium always exists.

Nash equilibrium



Properties of the underlying graph:

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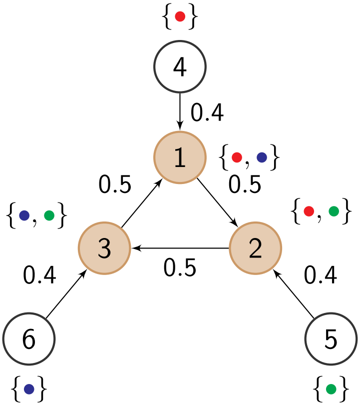

Properties of the underlying graph:

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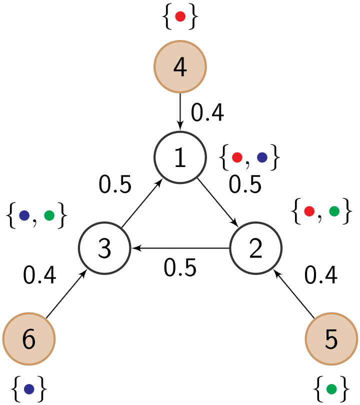

Properties of the underlying graph:

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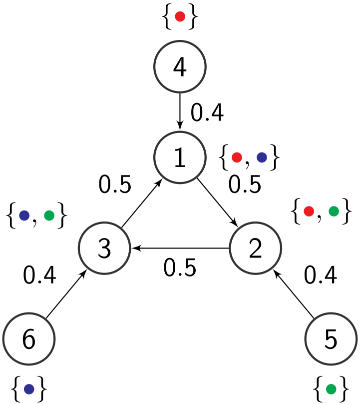

Properties of the underlying graph:

- Contains a cycle.
- Contains source nodes.

Question: Does Nash equilibrium always exist in social networks when the underlying graph

- is acyclic?
- has no source nodes?

Directed acyclic graphs

- A Nash equilibrium s is non-trivial if there is at least one player i such that $s_i \neq t_0$.

Directed acyclic graphs

Theorem. In a DAG, a non-trivial Nash equilibrium always exist.

Procedure to generate a non-trivial Nash equilibrium

Initialise: Assigns a product for each source node

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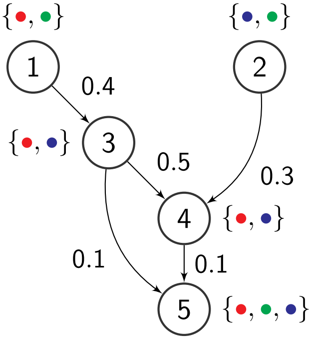

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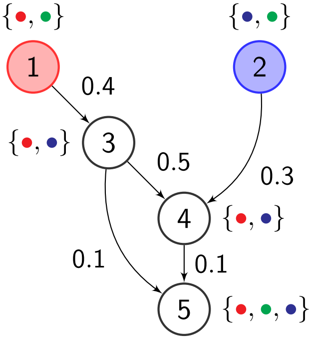

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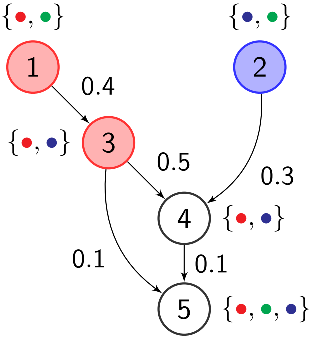

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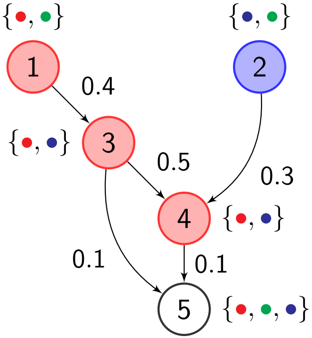

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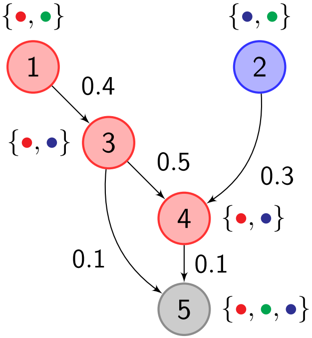

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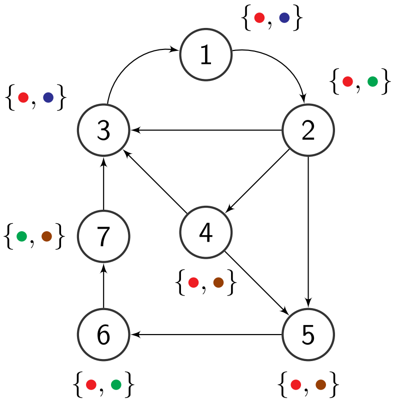
Theorem. A joint strategy s is a Nash equilibrium iff there is a run of the labelling procedure such that s is defined by the labelling function.

Graphs with no source nodes



"Circle of friends": everyone has a neighbour.

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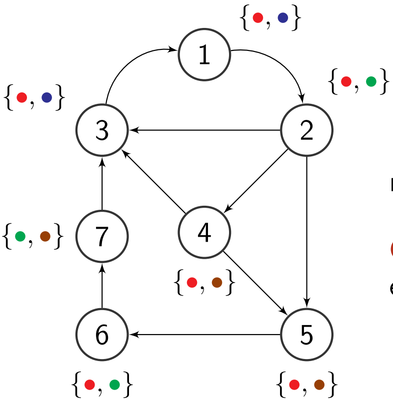

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Observation: $\overline{t_0}$ is always a Nash equilibrium.

Question: When does a non-trivial Nash equilibrium exist?

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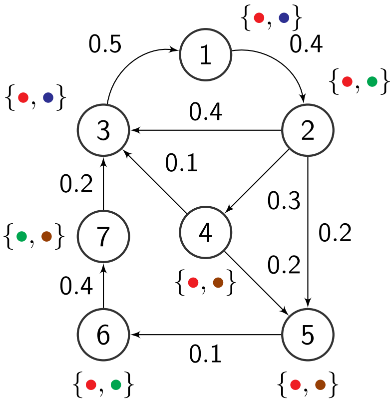

Threshold=0.3

Self sustaining subgraph

A subgraph C_t is self sustaining for product t if it is strongly connected and for all i in C_t,

- $t \in P(i)$
- $\sum_{j \in \mathcal{N}(i) \cap C_t} w_{ji} \geq \theta(i, t)$

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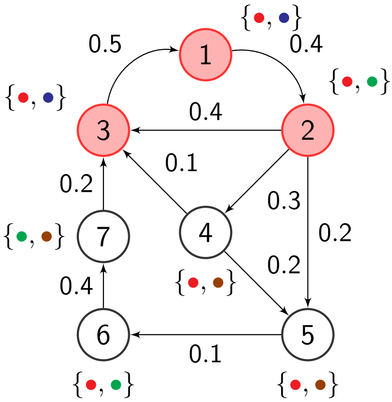

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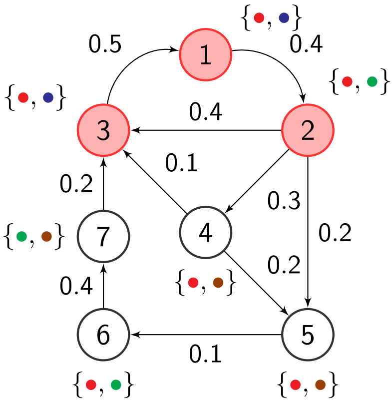

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Theorem. There is a non-trivial Nash equilibrium iff there exists a product t and a self sustaining subgraph C_t for t.

Graphs with no source nodes

For a product t,

- $X_t^0 := \{i \in V \mid t \in P(i)\}$
- $X_t^{m+1} := \{i \in V \mid \sum_{j \in \mathcal{N}(i) \cap X_j^m} w_{ji} \geq \theta(i, t)\}$
- $X_t := \bigcap_{m \in \mathbb{N}} X_t^m$

Theorem. There is a non-trivial Nash equilibrium iff there exists a product t such that $X_t \neq \emptyset$.

Finite Improvement Property

Fix a game.

- Profitable deviation: a pair (s, s') such that $s' = (s'_i, s_{-i})$ for some s'_i and $p_i(s') > p_i(s)$.
- Improvement path: a maximal sequence of profitable deviations.
- A game has the FIP if all improvement paths are finite.

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Theorem. Every two players social network game has the FIP.

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- A match: an element of ρ of the type $(\underline{t}, t)$ or $(t, \underline{t})$.
- Consider two successive matches in ρ.
- The corresponding segment of ρ is of one of the following types.
  Type 1. $(\underline{t}, t) \Rightarrow^* (\underline{t_1}, t_1)$.
  Type 2. $(\underline{t}, t) \Rightarrow^* (t_1, \underline{t_1})$.
  Type 3. $(t, \underline{t}) \Rightarrow^* (\underline{t_1}, t_1)$.
  Type 4. $(t, \underline{t}) \Rightarrow^* (t_1, \underline{t_1})$.

Proof, ctd

Type	p_1	p_2
1	increases by $> w_{21}$	decreases by $< w_{12}$
2, 3	increases	increases
4	decreases by $< w_{21}$	increases by $> w_{12}$

Table: Changes in p_1 and p_2

Proof, ctd

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- *Case 2.* $T_1 < T_4$.
  Then $p_2(\overline{t}) < p_2(\overline{t_1})$.
- Conclusion: $t \neq t_1$. So each match occurs in ρ at most once.

Proof, ctd

- So from some moment on in ρ no matches occur.

Proof, ctd

- So from some moment on in ρ no matches occur.
- So from that moment on the social welfare keeps increasing.

Proof, ctd

- So from some moment on in ρ no matches occur.
- So from that moment on the social welfare keeps increasing.
- Hence ρ is finite.

A generalization: two player coordination games

Theorem. Consider a finite two players game G such that

- $p_i(s) := f_i(s_i) + a_i(s_i = s_{-i})$,
  where $f_i : S_i \rightarrow \mathbb{R}$, $a_i > 0$ and

$$(s_i = s_{-i}) := \begin{cases} 1 & \text{if } s_i = s_{-i} \\ 0 & \text{otherwise} \end{cases}$$

Then G has the FIP.

Intuition: a_i is a bonus for player i for coordinating with his opponent.

Price of Anarchy and Price of Stability

Theorem. The price of anarchy and the price of stability for the games associated with the networks whose underlying graph is a DAG or a simple cycle is unbounded.

Proof

For a simple cycle.

Choose arbitrary $r > 0$ and ϵ such that $\epsilon < min(\frac{1}{4}, \frac{1}{2(r+1)})$.

Then $1 - 2\epsilon > 2\epsilon$ and $\frac{1-2\epsilon}{2\epsilon} > r$.

Consider the network

$$\begin{tikzcd} {}^{\{t_1,t_2\}}1 \arrow[r, bend left] & 2^{\{t_1,t_2\}} \arrow[l, bend left] \end{tikzcd}$$

Assume

$$w_{12} - \theta(2, t_2) = 1 - \epsilon,\ w_{21} - \theta(1, t_2) = -\epsilon,$$

$$w_{12} - \theta(2, t_1) = \epsilon,\ w_{21} - \theta(1, t_1) = \epsilon.$$

- Social optimum: (t_2, t_2) with social welfare $1 - 2\epsilon$.
- There are two Nash equilibria, (t_1, t_1) and (t_0, t_0) with the social welfare 2ϵ and 0.
- Price of anarchy: $\frac{1-2\epsilon}{0}$. We interpret it as ∞.
- Price of stability: $\frac{1-2\epsilon}{2\epsilon} > r$.