Point process

A point process X on $W \subset \mathbb{R}^2$ is a **random mechanism** for generating locally finite point patterns in W.

This week, we shall consider simple examples of such mechanisms:

- Uniformly random point;
- Binomial point process;
- Poisson process.

Finite point process

To define a finite point process, one must specify

- a probability mass function $(p_n)_{n\in\mathbb{N}_0}$ for the total number of points;

- a family of symmetric joint probability densities

$$j_n(x_1,\ldots,x_n),$$

$n \in \mathbb{N}$, on $(\mathbb{R}^2)^n$ for the locations of the points given that there are n of them.

Uniformly random point – definition

Let W be a bounded subset of $\mathbb{R}^2$ with area $|W| > 0$. Set

$$\begin{cases} p_1 & = & 1, \\ p_n & = & 0 \text{ for } n \neq 1, \end{cases}$$

and

$$j_1(x) = \frac{1}{|W|}.$$

Uniformly random point on rectangle

If $W = [0,a] \times [0,b]$, $a, b > 0$, is a rectangle, write (X, Y) for the coordinates.

Then, for $x \in [0,a]$, $y \in [0,b]$,

$$\mathbb{P}(X \leq x; Y \leq y) = \int_0^x \int_0^y \frac{1}{ab} du dv = \frac{x}{a}\frac{y}{b}$$

$$= \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y).$$

Hence X and Y are independent and uniformly distributed on, respectively, $[0,a]$ and $[0,b]$.

Uniformly random point on disc

If $W = B(0,t)$, $t > 0$, is a disc, write (R, Φ) for the polar coordinates.

Then, for $r \in [0,t]$, $\phi \in [0, 2\pi)$,

$$\mathbb{P}(R \le r; \Phi \le \phi) = \int_0^r \int_0^\phi \frac{1}{\pi t^2} s ds d\psi = \frac{r^2}{t^2} \frac{\phi}{2\pi}$$

$$= \mathbb{P}(R \le r)\mathbb{P}(\Phi \le \phi).$$

Hence R and Φ are independent, Φ is uniformly distributed on $[0, 2\pi)$ and R^2 is uniformly distributed on $[0, t^2]$.

Uniformly random point – restriction property

Suppose that $B \subset A$, $0 < |B|$, $|A| < \infty$, and consider uniformly random points X_A on A and X_B on B. Then

$$\mathbb{P}(X_A \in C | X_A \in B) = \mathbb{P}(X_B \in C)$$

for all $C \subset B$.

Proof: Note that

$$\mathbb{P}(X_A \in D) = \int_D \frac{1}{|A|} dx = \frac{|D|}{|A|}$$

for all $D \subset A$. Hence for $C \subset B \subset A$,

$$\begin{aligned} \mathbb{P}(X_A \in C | X_A \in B) &= \frac{\mathbb{P}(X_A \in C; X_A \in B)}{\mathbb{P}(X_A \in B)} \\ &= \frac{|C \cap B|/|A|}{|B|/|A|} = \frac{|C|}{|B|} = \mathbb{P}(X_B \in C). \end{aligned}$$

Uniformly random point – composition property

Suppose that $A = \bigcup_{i=1}^{n} A_i$ and consider a uniformly random point X_A on A.

Then

$$\mathbb{P}(X_A \in A_i) = \frac{|A_i|}{|A|}.$$

Simulation

In **spatstat**, the command

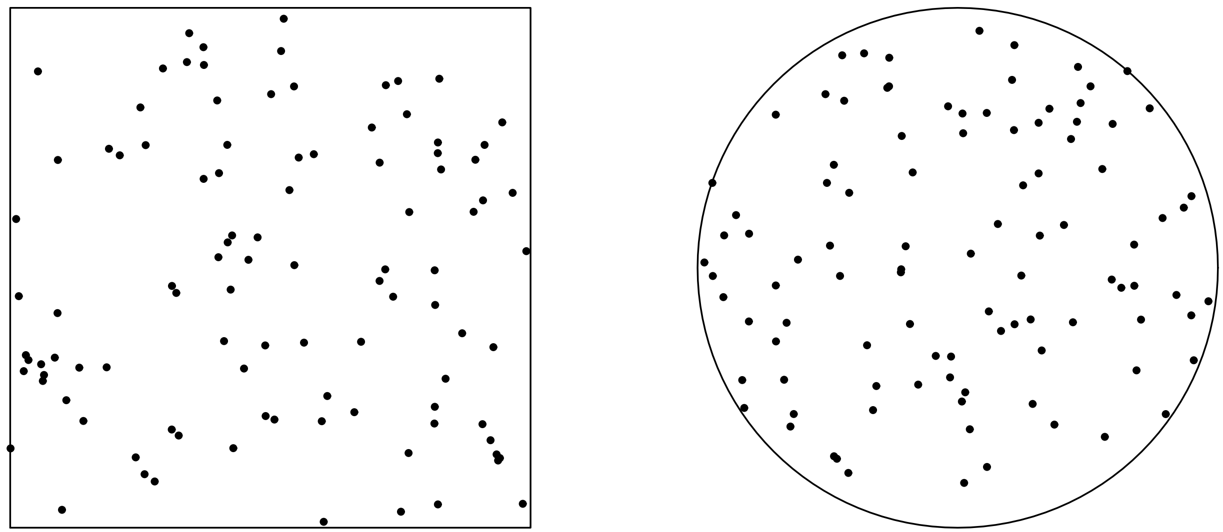
```
runifpoint(n, win, nsim=1)
```

generates `n` independent uniformly random points in the window `win`.

As we saw, this is easy for **circular** and **rectangular** windows. For complex windows, the function samples in a rectangle that contains the window and uses the **restriction** property.

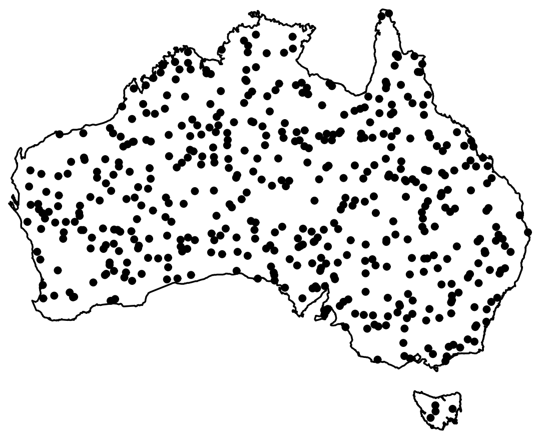
For windows that consist of several connected components, use the **composition** property.

Examples – rectangle and disc



$n = 100$

Examples – complex window



```
XA <- runifpoint(n=500, win=austates, nsim=1)
```

Binomial point process

Let W be a bounded subset of $\mathbb{R}^2$ with area $|W| > 0$. Let, for fixed $n \in \mathbb{N}_0$, $X_1, \ldots, X_n$ be n independent uniformly random points. Then

$$X = \{X_1, \ldots, X_n\}$$

is the **binomial point process** with

$$\begin{cases} p_n &= 1, \\ p_m &= 0 \text{ for } m \neq n, \end{cases}$$

and

$$j_n(x_1, \ldots, x_n) = \left(\frac{1}{|W|} \right)^n .$$

Binomial point process – void probability

For $A \subset W$ and $k \in \{0, \ldots, n\}$,

$$\mathbb{P}(N_X(A) = k) = \binom{n}{k} \left(\frac{|A|}{|W|}\right)^k \left(1 - \frac{|A|}{|W|}\right)^{n-k}$$

since a uniformly random point X_i has probability $|A|/|W|$ of falling in A.

In particular, for $k = 0$, we obtain the **void probability**

$$v(A) = \left(1 - \frac{|A|}{|W|}\right)^n .$$

Binomial point process – intensity function

Let X be a binomial point process with n points on W. Then, for $A \subset W$,

$$\mathbb{E}N_X(A) = \frac{n|A|}{|W|} = \int_A \frac{n}{|W|}\,dx.$$

Hence the **intensity function**

$$\rho^{(1)}(x) = \frac{n}{|W|}$$

does not depend on $x \in W$.

Binomial point process – 2nd moment measure

Let $X = \{X_1, \ldots, X_n\} \subset W$ be a binomial point process.

Then, for $A, B \subset W$,

$$\mathbb{E}\left[N_X(A)N_X(B)\right] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} 1\{X_i \in A; X_j \in B\}\right]$$

$$= \sum_{i=1}^{n}\sum_{j\neq i} \mathbb{P}(X_i \in A; X_j \in B) + \sum_{i=1}^{n} \mathbb{P}(X_i \in A \cap B).$$

Since the X_i are independent,

$$\mu^{(2)}(A \times B) = n(n-1)\frac{|A||B|}{|W|^2} + n\frac{|A \cap B|}{|W|}.$$

Binomial point process – pair correlation function

The second order factorial moment measure

$$\alpha^{(2)}(A \times B) = \mu^{(2)}(A \times B) - \alpha^{(1)}(A \cap B)$$

equals

$$n(n-1)\frac{|A||B|}{|W|^2} = \int_A \int_B \frac{n(n-1)}{|W|^2}\, dxdy$$

hence

$$\rho^{(2)}(x, y) = \frac{n(n-1)}{|W|^2}$$

and

$$g(x, y) = \frac{\rho^{(2)}(x, y)}{\rho^{(1)}(x)\rho^{(1)}(y)} = \frac{n-1}{n} < 1.$$

CSR – Complete Spatial Randomness

A simple point process satisfies CSR if the following hold:

1. The points have no preference for any spatial location (**homogeneity**).

2. Information about the outcome in one region has no influence on that in other regions (**independence**).

Binomial point process and CSR

The binomial point process X does not satisfy the **independence** assumption. Indeed, for $n \neq 0$ and $A \subset W$ such that $0 < |A| < |W|$,

$$\mathbb{P}(N_X(A) = n; N_X(W \setminus A) = 0) = \left(\frac{|A|}{|W|}\right)^n$$

but

$$\mathbb{P}(N_X(A) = n)\mathbb{P}(N_X(W \setminus A) = 0) = \left(\frac{|A|}{|W|}\right)^{2n}.$$

Limit of binomial point processes

Let $B_n \subset \mathbb{R}^2$ be a series of growing balls centred at the origin such that $n/|B_n| \equiv \lambda$ is constant ($0 < \lambda < \infty$).

Then any bounded set A is covered by B_n for n sufficiently large, and, in this case, for $k \leq n$,

$$\mathbb{P}^{(n)}(N(A) = k) = \binom{n}{k} \left(\frac{|A|}{|B_n|} \right)^k \left(1 - \frac{|A|}{|B_n|} \right)^{n-k},$$

writing $\mathbb{P}^{(n)}$ for the distribution of the binomial point process of n points in B_n.

Claim: For fixed $k \in \mathbb{N}_0$,

$$\mathbb{P}^{(n)}(N(A) = k) \to e^{-\lambda|A|}(\lambda|A|)^k / k!$$

as $n \to \infty$.

Proof: If $n \geq k$ and $A \subset B_n$, then

$$\binom{n}{k}\left(\frac{|A|}{|B_n|}\right)^k\left(1-\frac{|A|}{|B_n|}\right)^{n-k} =$$

$$\frac{|A|^k}{k!} \times \frac{n}{|B_n|}\frac{n-1}{|B_n|}\cdots\frac{n-k+1}{|B_n|} \times \left(1-\frac{|A|}{|B_n|}\right)^{n-k}.$$

Note that

$$\frac{n}{|B_n|} \to \lambda; \quad \frac{n-k+1}{|B_n|} \to \lambda$$

and

$$\left(1-\frac{|A|}{|B_n|}\right)^{n-k} = \left(1-\frac{|A|}{|B_n|}\right)^{|B_n|(n-k)/|B_n|} \to \exp[-\lambda|A|].$$

Limit of binomial point processes (ctd)

If $A \cap B = \emptyset$, for $k, l \in \mathbb{N}_0$, take $n \geq k + l$ so large that $A \cup B \subset B_n$. Then

$$\mathbb{P}^{(n)}(N(A) = k; N(B) = l) =$$

$$\binom{n}{k} \left(\frac{|A|}{|B_n|}\right)^k \binom{n-k}{l} \left(\frac{|B|}{|B_n|}\right)^l \left(1 - \frac{|A \cup B|}{|B_n|}\right)^{n-k-l}$$

which, as $n \to \infty$, tends to

$$e^{-\lambda|A|} \frac{(\lambda|A|)^k}{k!} e^{-\lambda|B|} \frac{(\lambda|B|)^l}{l!}.$$

In the limit, $N_X(A)$ and $N_X(B)$ are **independent**.

The homogeneous Poisson process

A point process X on $\mathbb{R}^2$ is a **homogeneous Poisson process** with **intensity** $\lambda > 0$ if

- $N_X(A)$ is Poisson distributed with mean $\lambda|A|$ for every bounded set $A \subset \mathbb{R}^2$;

- for any k disjoint bounded sets $A_1, \ldots, A_k$, $k \in \mathbb{N}$, the random variables $N_X(A_1), \ldots, N_X(A_k)$ are independent.

The process therefore satisfies **CSR**.

Poisson process – conditionally independent points

Let X be a homogeneous Poisson process on $\mathbb{R}^2$ with intensity $\lambda > 0$ and $A \subset \mathbb{R}^2$ bounded with $|A| > 0$.

Claim: Conditionally on the event $\{N_X(A) = n\}$, X restricted to A is a binomial point process of n points.

Proof

Clearly, conditionally on $N_X(A) = n$, $p_n = 1$ and $p_m = 0$ for $m \neq n$.

Fix $x_1, \ldots, x_n \in A$ and set $B_i = B(x_i, \epsilon)$ for ϵ small enough to make the B_i disjoint. Then

$$\mathbb{P}(N_X(B_1) = 1; \cdots ; N_X(B_n) = 1 | N_X(A) = n)$$

$$= \frac{\mathbb{P}(N_X(B_1) = 1; \cdots ; N_X(B_n) = 1; N_X(A \setminus \cup B_i) = 0)}{\mathbb{P}(N_X(A) = n)}.$$

Using the two defining properties, we get

$$\frac{\lambda |B_1| e^{-\lambda |B_1|} \cdots \lambda |B_n| e^{-\lambda |B_n|} e^{-\lambda |A \setminus \cup B_i|}}{\lambda^n |A|^n e^{-\lambda |A|} / n!} = \frac{n!}{|A|^n} \prod_{i=1}^{n} |B_i|.$$

Upon dividing by $n!$, the number of permutations of n points, in an infinitesimal sense, the probability that X_1 falls in dx_1, X_2 in $dx_2, \ldots$ is

$$j_n(x_1, \ldots, x_n | N_X(A) = n) dx_1 \ldots dx_n = \frac{1}{|A|^n} dx_1 \ldots dx_n,$$

the scatter density of a binomial point process on A.

The homogeneous Poisson process – properties

Since $N_X(A)$ is Poisson distributed with mean $\lambda|A|$, the **void probability** of A is

$$v(A) = \exp\left[-\lambda|A|\right]$$

and

$$\mathbb{E}N_X(A) = \int_A \lambda dx.$$

The **intensity function**

$$\rho^{(1)}(x) = \lambda$$

is constant.

Poisson process – 2nd order moment measure

Let X be a homogeneous Poisson process on $\mathbb{R}^2$.

For bounded $A, B \subset \mathbb{R}^2$, conditionally on $N_X(A \cup B) = n$,

$\mathbb{E}\left[N_X(A)N_X(B)|N_X(A \cup B) = n\right]$

$$= n(n-1)\frac{|A||B|}{|A \cup B|^2} + n\frac{|A \cap B|}{|A \cup B|}.$$

Recalling $\mathbb{E}N_X(A \cup B) = \lambda|A \cup B|$ and

$$\mathbb{E}\left[N_X(A \cup B)(N_X(A \cup B) - 1)\right] = (\lambda|A \cup B|)^2$$

we get

$$\mu^{(2)}(A \times B) = \lambda^2|A||B| + \lambda|A \cap B|.$$

Poisson process – pair correlation function

The second order factorial moment measure

$$\alpha^{(2)}(A \times B) = \mu^{(2)}(A \times B) - \alpha^{(1)}(A \cap B)$$

equals

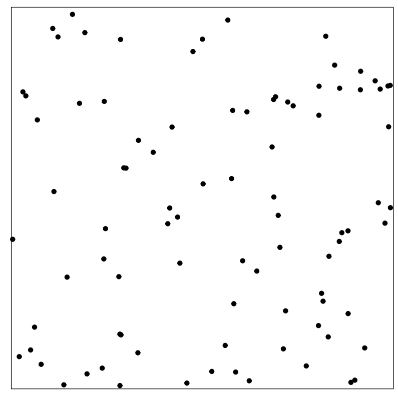
$$\int_A \int_B \lambda^2 dx dy$$

hence $\rho^{(2)}(x, y) = \lambda^2$ and

$$g(x, y) = 1$$

regardless of x, y.

Simulation



```
rpoispp(lambda=100, win=square(1), nsim=1)
```

Monte Carlo testing

To test whether the points of data pattern $\mathbf{x} = \{x_1, \ldots, x_m\} \subset W$ are scattered independently and uniformly:

- choose some statistic V;
- calculate $V_1 = V(X_1), \ldots, V_{n-1} = V(X_{n-1})$ for $n-1$ independent binomial point processes $X_1, \ldots, X_{n-1}$ on W, each having m points;
- calculate $V_n = V(\mathbf{x})$;
- if V_n is among the k most extreme values, reject the null hypothesis.

Monte Carlo testing – remarks

- The level of this test is k/n.
- The procedure is valid since under the null hypothesis any permutation of the V_i is equally likely.
- The test conditions on the number of points.
- One-sided tests for clustering/inhibition, or two-sided tests both apply.
- The test may be conservative, that is, have low power (Myllymäki et al., 2017).

Point process statistics – K-function

For $r \geq 0$,

$$K(r) = \int_{B(0,r)} g(0,z)dz.$$

Under the null hypothesis, for r such that $B(0,r) \subset W$,

$$K(r) = \int_{B(0,r)} \frac{m-1}{m} dz = \frac{m-1}{m} \pi r^2;$$

for a Poisson process with intensity λ

$$K(r) = \int_{B(0,r)} 1 dz = \pi r^2.$$

Estimating the K-function

Let X be a stationary point process, $\mathbf{x}$ a realisation in W. Then

$$\widehat{\lambda^2 K(r)} = \widehat{\lambda^2 K(r; \mathbf{x})} = \sum_{x \in \mathbf{x}} \sum_{y \in \mathbf{x}}^{\neq} \frac{1\{||y - x|| \leq r\}}{|W \cap W_{y-x}|}$$

is an unbiased estimator of $\lambda^2 K(r)$.

For a homogeneous Poisson process,

$$\mathbb{E}\left[\frac{N_X(W)(N_X(W) - 1)}{|W|^2}\right] = \frac{\alpha^{(2)}(W \times W)}{|W|^2} = \lambda^2$$

so

$$\frac{n(\mathbf{x})(n(\mathbf{x}) - 1)}{|W|^2}$$

is an unbiased estimator for λ^2.

Proof

If X is stationary, by the Campbell–Mecke formula,

$$\mathbb{E}\left[\sum_{x\in X\cap W}\sum_{y\in X\cap W}^{\neq}\frac{1\{||y-x||\leq r\}}{|W\cap W_{y-x}|}\right]=\int_W\int_W\frac{1\{||y-x||\leq r\}}{|W\cap W_{y-x}|}\rho^{(2)}(x,y)dxdy.$$

By stationarity, $\rho^{(2)}(x,y)=\rho^{(2)}(0,y-x)$ and substituting $z=y-x$ gives

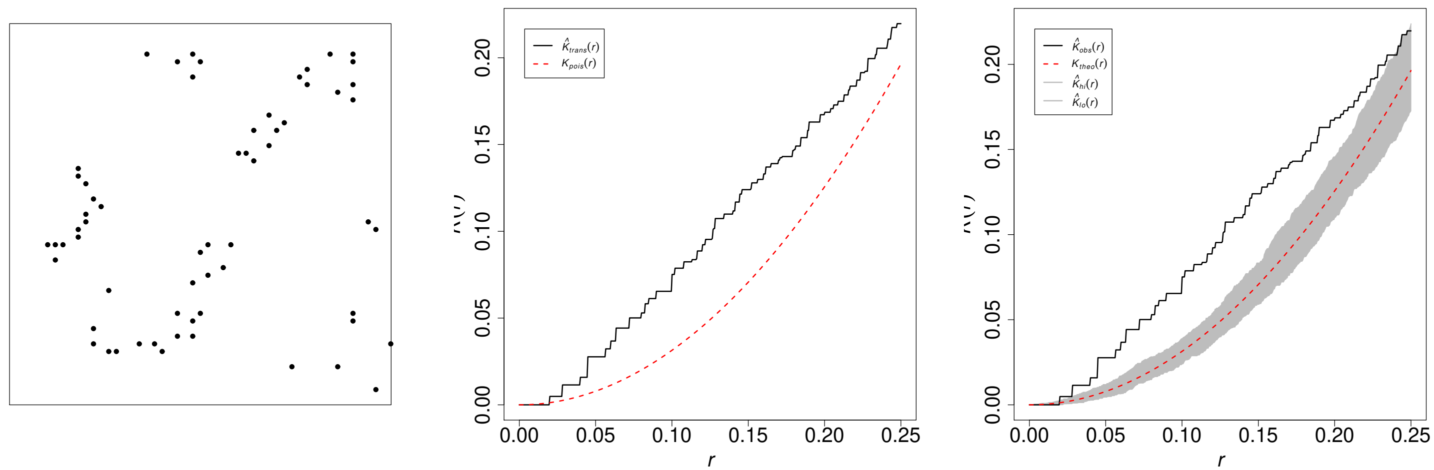
$$\int_W\left(\int_{W-x}\frac{1\{||z||\leq r\}}{|W\cap W_z|}\rho^{(2)}(0,z)dz\right)dx$$

which is equal to

$$\int_{B(0,r)}\left(\int_{W\cap W_{-z}}\frac{1}{|W\cap W_z|}dx\right)\rho^{(2)}(0,z)dz=\int_{B(0,r)}\lambda^2 g(0,z)dz=\lambda^2K(r)$$

after changing the order of integration.

Example: Redwood data




```
Kest(redwood, correction="translate")
envelope(redwood, fun=Kest, correction="translate",
   nsim=99, nrank=5)
```

Note: two.sided so level of pointwise test is 10/100.

Point process statistics – F-function

Define the **empty space function** as

$$F(r) = \mathbb{P}(d(0, X) \leq r), \quad r \geq 0.$$

If X is stationary, the definition does not depend on the choice of origin.

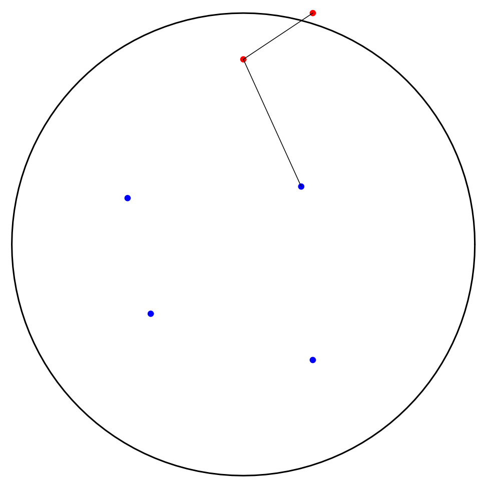
Under the null hypothesis, for r such that $B(0, r) \subset W$,

$$F(r) = 1 - v(B(0, r)) = 1 - \left(1 - \frac{\pi r^2}{|W|}\right)^m;$$

for a Poisson process with intensity λ

$$F(r) = 1 - v(B(0, r)) = 1 - \exp\left[-\lambda \pi r^2\right].$$

Estimating empty space function – edge effects



In general, $d(l, X) \neq d(l, X \cap W)$.

Estimating the empty space function

Let X be a stationary point process, $\mathbf{x}$ a realisation in W, and $L \subset W$ a lattice of points. Write $d(\cdot,\cdot)$ for the Euclidean distance.

For $r \geq 0$,

$$\widehat{F(r)} = \widehat{F(r;\mathbf{x})} = \sum_{l \in L} \frac{1\{d(l,\mathbf{x}) \leq r; d(l,W^c) \geq d(l,\mathbf{x})\}}{\#\{\tilde{l} \in L : d(\tilde{l},W^c) \geq d(l,\mathbf{x})\}}$$

$(0/0 = 0)$ is an unbiased estimator (Chiu and Stoyan, 1998).

Crucial observation:

$$d(l, X \cap W) \leq d(l, W^c) \Rightarrow d(l, X \cap W) = d(l, X).$$

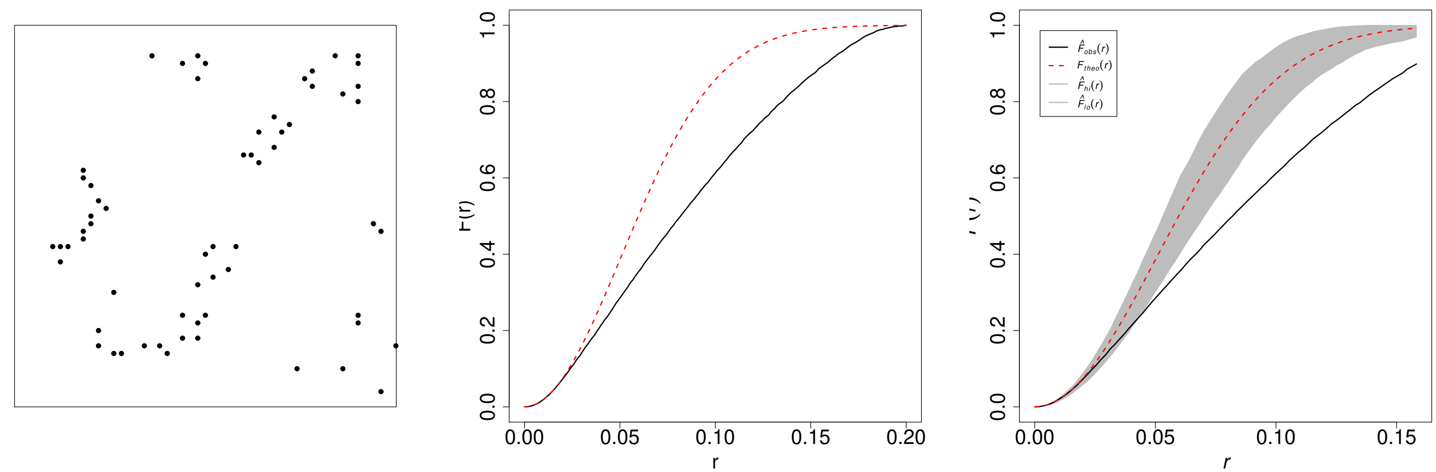
Proof

$$\mathbb{E}\left[\sum_{l\in L}\frac{1\{d(l,X\cap W)\le r; d(l,W^c)\ge d(l,X\cap W)\}}{\#\{\tilde{l}\in L: d(\tilde{l},W^c)\ge d(l,X\cap W)\}}\right]$$

$$=\sum_{l\in L}\mathbb{E}\left[\frac{1\{d(l,X)\le r; d(l,W^c)\ge d(l,X\cap W)=d(l,X)\}}{\#\{\tilde{l}\in L: d(\tilde{l},W^c)\ge d(l,X)\}}\right]$$

$$=\sum_{l\in L}\int_0^r\frac{1\{d(l,W^c)\ge s\}}{\#\{\tilde{l}\in L: d(\tilde{l},W^c)\ge s\}}dF(s)=F(r).$$

Example: Redwood data




```
Fest(redwood, correction="cs", r=seq(0,0.2,by=0.001))
envelope(redwood, fun=Fest, correction="cs",
   nsim=99, nrank=5)
```

Deviation test

The K- or F-function may be used in a Monte Carlo test.

- $V(\mathbf{x}) = \widehat{K}(r_0; \mathbf{x})$ for a **fixed** $r_0 > 0$;

- $V(\mathbf{x}) = \sup_{r \le r_0} |\widehat{K(r; \mathbf{x})} - \pi r^2|$

or

$$V(\mathbf{x}) = \int_0^{r_0} \{\widehat{K(r; \mathbf{x})} - \pi r^2\}^2 \, dr.$$

Max absolute deviation test and global envelopes

The Monte Carlo test with

$$V(\mathbf{x}) = \sup_{r \leq r_0} |\widehat{K(r; \mathbf{x})} - \pi r^2|$$

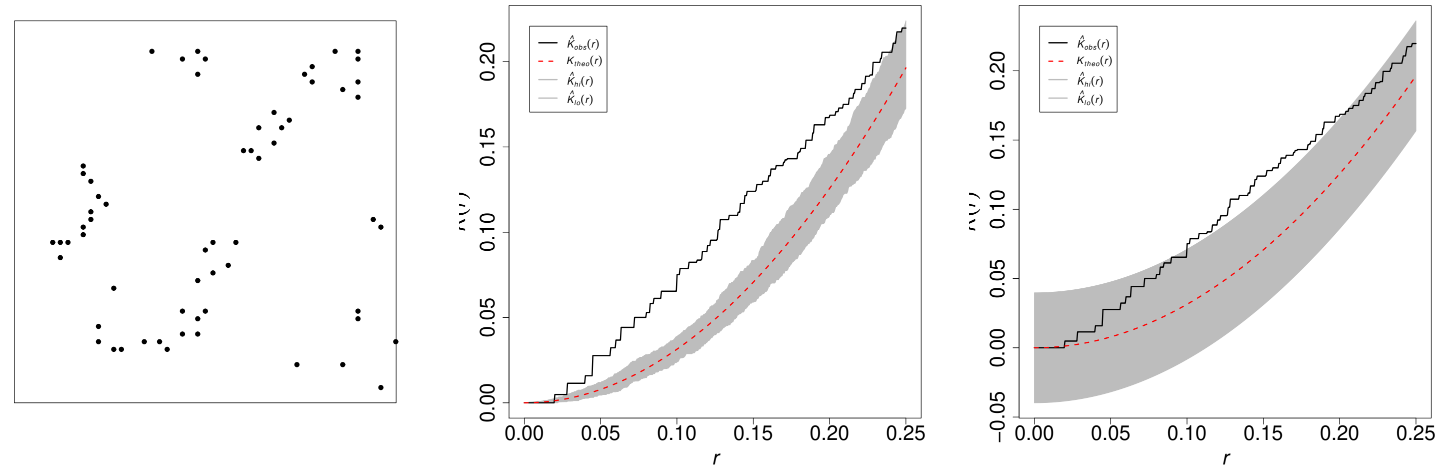
is implemented in **spatstat:**

```
mad.test(redwood, fun=Kest, correction="translate",
   nsim=19)
```

A graphical illustration is to plot envelopes with a constant width equal to twice max$\{V(X_i) : i = 1, \ldots, n-1\}$:

```
envelope(redwood, fun=Kest, correction="translate",
   nsim=99, nrank=5, global=TRUE)
```

Example: Redwood data



Middle: local envelope; Right: global envelope.

CSR is rejected at level `nrank` / (`nsim` + 1) = 5%.

Inhomogeneous Poisson process

Suppose that counts in disjoint sets are independent and for every bounded set $A \subset \mathbb{R}^2$, $N_X(A)$ is Poisson distributed with mean

$$\Lambda(A) = \int_A \lambda(x)dx$$

for some integrable function $\lambda : \mathbb{R}^2 \to [0, \infty)$.

Then X is a Poisson process with **intensity function** λ.

Inhomogeneous Poisson process – moments

Let X be a Poisson process with intensity function $\lambda(\cdot)$. Then

$$\rho^{(1)}(x) = \lambda(x)$$

and

$$\rho^{(2)}(x, y) = \lambda(x)\lambda(y).$$

Consequently

$$g(x, y) = 1.$$

Proof

Since $N_X(A)$ is Poisson distributed with mean $\Lambda(A)$,

$$\alpha^{(1)}(A) = \mathbb{E}N_X(A) = \Lambda(A) = \int_A \lambda(x)dx,$$

and $\rho^{(1)}(x) = \lambda(x)$.

For $A, B \subset \mathbb{R}^2$,

$$\mu^{(2)}(A \times B) = \mathbb{E}\left[N_X(A)N_X(B)\right] =$$
$$\mathbb{E}\left[\{N_X(A \cap B) + N_X(A \setminus B)\}\{N_X(A \cap B) + N_X(B \setminus A)\}\right].$$

Since counts in disjoint sets are independent and Poisson distributed, $\mu^{(2)}(A \times B)$ is equal to

$$\mathbb{E}\left[N_X(A \cap B)^2\right] + \Lambda(A \cap B)\Lambda(B \setminus A) + \Lambda(A \setminus B)\left[\Lambda(A \cap B) + \Lambda(B \setminus A)\right]$$

$$= \Lambda(A \cap B)^2 + \Lambda(A \cap B) + \Lambda(A \cap B)\Lambda(B \setminus A) + \Lambda(A \setminus B)\Lambda(B)$$
$$= \Lambda(A \cap B)\Lambda(B) + \Lambda(A \cap B) + \Lambda(A \setminus B)\Lambda(B) = \Lambda(A)\Lambda(B) + \Lambda(A \cap B).$$

The claim follows upon recalling that

$$\mu^{(2)}(A \times B) = \alpha^{(2)}(A \times B) + \alpha^{(1)}(A \cap B).$$

Example: Linear trend

$$\lambda(x, y) = 250x, \quad (x, y) \in [0, 1]^2$$

is implemented by

```
lambda1 <- function(x,y)
{
   return( 250 * x )
}
```

Example: Radial trend

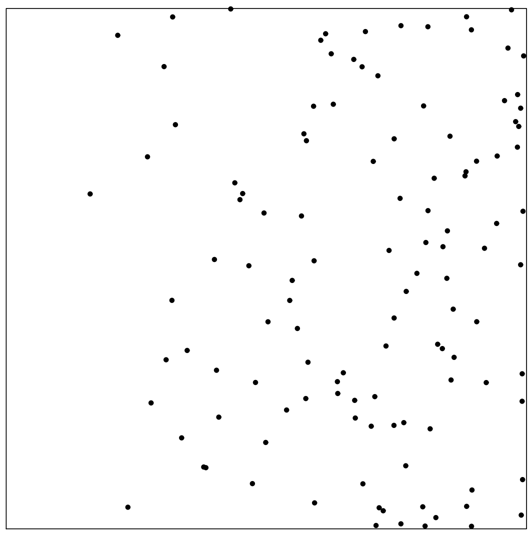
$$\lambda(x,y) = 1000\left(1/2 - \sqrt{(x-1/2)^2 + (y-1/2)^2}\right),$$

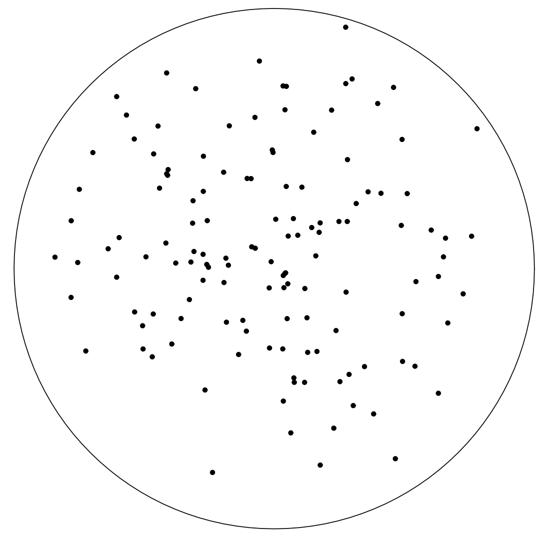
$$(x,y) \in \{(u,v) \in \mathbb{R}^2 : (u-1/2)^2 + (v-1/2)^2 \leq 1/4\}$$

is implemented by

```
lambda2 <- function(x,y)
{
  dist <- sqrt( ( x - 0.5 )^2 + ( y - 0.5 )^2 )
  return ( 1000 * ( 0.5 - dist ) )
}
```

Examples





Implementation

```
X1 <- rpoispp(lambda=lambda1, lmax=250,
              win=owin(c(0,1), c(0,1)))

X2 <- rpoispp(lambda=lambda2, lmax=500,
   win=disc(radius=1/2, centre=c(1/2, 1/2)))
```

`lmax`, the maximal value of the intensity function is optional, but results in faster simulation.

Instead of a function, a pixel image may be used for `lambda`.

Poisson process – thinning property

Let X be a homogeneous Poisson process on the bounded window W with intensity L and λ a function such that

$$0 \leq \lambda(x, y) \leq L.$$

Construct the point process Y by retaining each point of X with probability

$$\frac{\lambda(x, y)}{L}$$

independently of other points.

Claim: Y is a Poisson process with intensity function λ.

Proof

Counts in disjoint sets are independent by construction.

We show that $N_Y(A)$ is Poisson distributed by conditioning on $N_X(A)$. Indeed, since $N_X(A)$ is Poisson distributed with mean $L|A|$, and conditionally on $N_X(A) = k$, the points of X are uniformly scattered in A before being subjected to thinning,

$$\mathbb{P}(N_Y(A) = n) =$$

$$= \sum_{k=n}^{\infty} \frac{e^{-L|A|}(L|A|)^k}{k!} \frac{1}{|A|^k} \binom{k}{n} \int_A \cdots \int_A \prod_{i=1}^{n} \frac{\lambda(x_i)}{L} \prod_{i=n+1}^{k} \left(1 - \frac{\lambda(x_i)}{L}\right) dx_1 \ldots dx_k$$

$$= \sum_{k=n}^{\infty} \frac{e^{-L|A|}}{k!} \binom{k}{n} (L|A| - \Lambda(A))^{k-n} \Lambda(A)^n = e^{-L|A|} \frac{\Lambda(A)^n}{n!} e^{L|A| - \Lambda|A|},$$

where

$$\Lambda(A) = \int_A \lambda(x) dx.$$

References

1. J. Besag and P.J. Diggle. Simple Monte Carlo tests for spatial pattern. *Applied Statistics* 26:327–333, 1977.

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3. M. Myllymäki, T. Mrkvička, P. Grabarnik, H. Seijo and U. Hahn. Global envelope tests for spatial processes. *Journal of the Royal Statistical Society* 79:381–404, 2017.

4. J.F.C. Kingman. Poisson processes. Clarendon, 1993.

Assessment

The `R`-package **spatstat** contains a number of mapped patterns of pine trees.

For each of `finpines`, `japanesepines`, `longleaf` and `swedishpines`, test whether the CSR assumptions are satisfied.

Discuss and interpret your results!