# Perfect simulation for length-interacting polygonal Markov fields in the plane

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# Polygonal configurations

Let $D \subset \mathbb{R}^2$ be bounded, open and convex with piece-wise smooth boundary $\partial D$ and area $|D| > 0$.

Let the family $\Gamma_D$ of admissible polygonal configurations in $D$ consist of all planar graphs $\gamma$ in $D \cup \partial D$ with line segments as edges such that:

- all interior vertices of $\gamma$ (in $D$) have degree $2$;
- $\gamma \cap \partial D$ is empty or consists of vertices of degree $1$;
- the edges of $\gamma$ do not intersect;
- no two edges of $\gamma$ are co-linear.

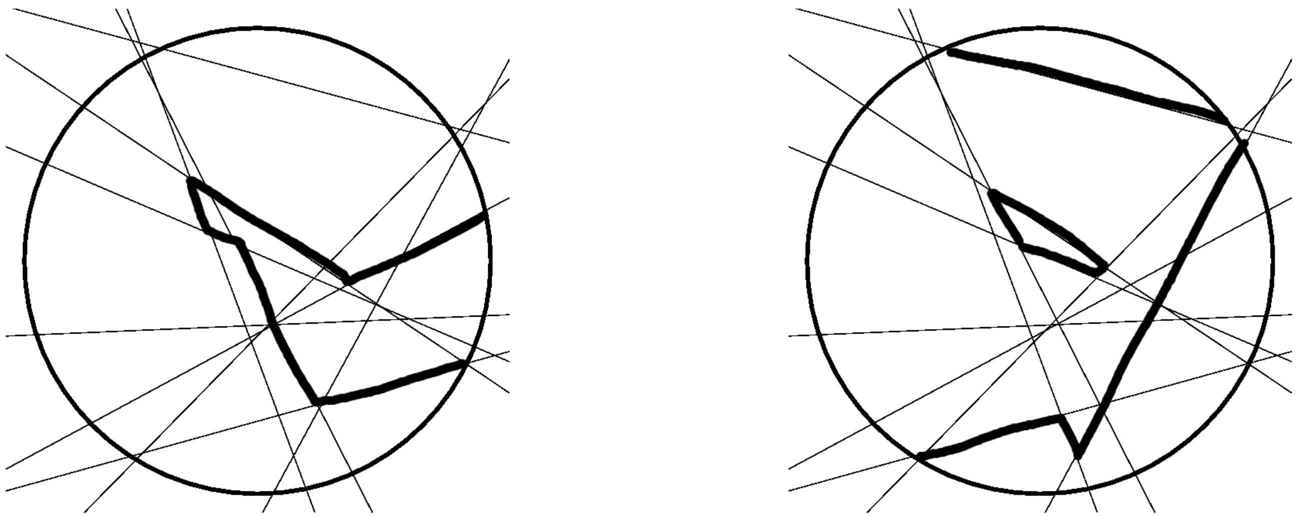
# *Arak model*

(Arak, 1982)

Use the **Poisson line process** $\Lambda$ as a **skeleton**:

- use each line exactly once;
- weigh by $\exp\left[-2\ell(\gamma)\right]$.

Note there may be multiple or no admissible polygonal configurations $\gamma$ for a given skeleton.



# Notes

The Arak model $\mathcal{A}_D$ (with free boundary conditions) is

- stationary and isotropic;
- consistent: $\mathcal{A}_D$ is equal in distribution to the restriction of $\mathcal{A}_{D'}$ to $D$ for $D' \supseteq D$;
- Poisson line transects of rate $2$ identical to those of $\Lambda$;
- Markov: the conditional distribution of the field inside a piece-wise smooth closed curve depends only on the intersection points and angles;
- partition function is known in closed form ($e^{\pi|D|}$).

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# Length-interacting Arak process $\mathcal{A}_D^{[\beta]}$

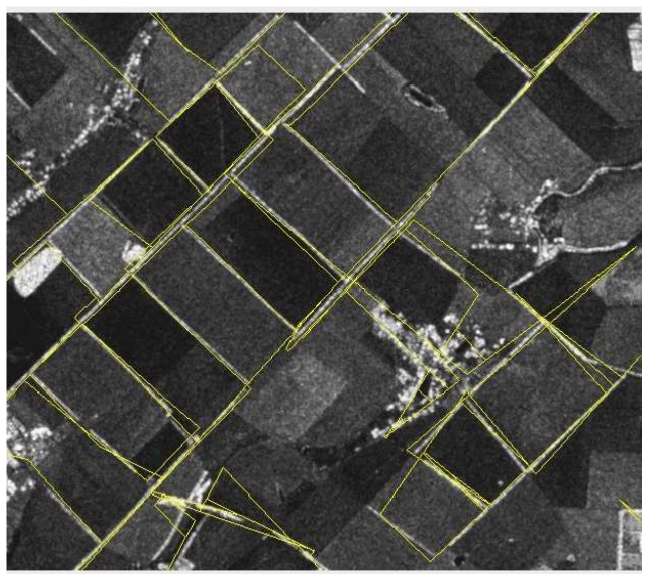
is defined by density

$$f(\gamma) \propto \exp(-\beta \ell(\gamma)), \quad \beta > 0,$$

with respect to $\mathcal{L}(\mathcal{A}_D)$, the law of the Arak process.

**Interpretation:**

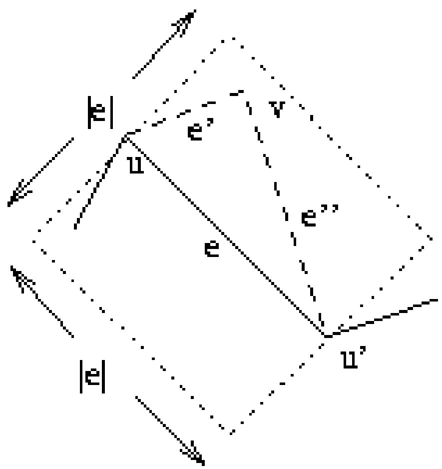
- configurations with small $\ell(\gamma)$ are favoured;
- can be used as priors in image analysis (segmentation, network extraction).

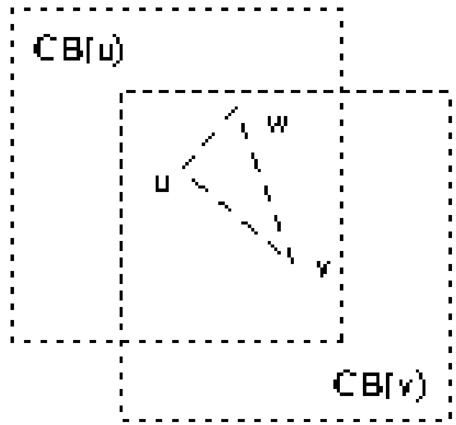


# Metropolis–Hastings sampling

**Idea:** generate a Markov chain that converges to $\mathcal{L}(\mathcal{A}_D^{[\beta]})$ by proposing a new state and accepting it in compliance with the detailed balance equations.

Clifford and Nicholls (1994) propose 6 transition types. For example, new vertices are created by the addition of a new triangle, but also by splitting an existing edge.



# *Disadvantages*

- very complicated book keeping;
- rate of convergence unknown;
- slow in practice;
- prone to bugs.

**Goal:** reformulate as a hard object process and apply exact simulation methods developed in the latter context.

# *Hard object processes*

have realisations

$$\mathbf{y} = \{y_1 = (x_1, m_1), \ldots, y_n = (x_n, m_n)\}, \quad n \in \mathbb{N}_0,$$

with $x_i \in D$ and $m_i$ in some mark space $M$ that parametrises objects $Z(m_i)$ such that

$$[x_i + Z(m_i)] \cap [x_j + Z(m_j)] = \emptyset$$

for all $\mathbf{y}$ having positive probability.

(Ripley & Kelly, 1977; Van Lieshout, 2000)

**Polygonal contours:** identify contour $\theta$ with its left-most point $\iota[\theta]$ and set

$$M = \{\theta \in \mathcal{C} \mid \iota[\theta] = 0\}$$

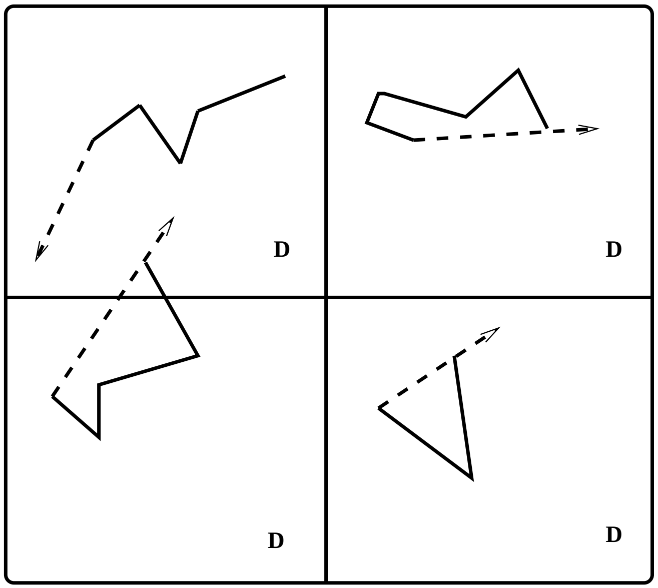
for suitable $\mathcal{C}$ and $\emptyset$ included by convention.

# *Empty boundary condition*

Let $\mathcal{C}_{(-n,n)^2}$ be the set of all closed polygonal contours in $(-n,n)^2$ which do not touch $\partial(-n,n)^2$. Then

$$\mathcal{C} = \cup_{n=1}^{\infty} \mathcal{C}_{(-n,n)^2}.$$

The mark distribution $\Theta_*^{[\beta]}$ on $M$ can, for $\beta \geq 2$, be defined by a **random walk representation** $Z_t$.



# *Random walk representation*

(Schreiber, 2006)

1. From $Z_0 = 0$,
   - set initial direction uniformly in $(0, 2\pi)$;
   - between updates, move in constant direction with speed $1$;
   - with intensity $4$, update direction by choosing angle $\phi \in (0, 2\pi)$ between old and new direction according to $|\sin(\phi)|/4$.
2. $Z_t$ is killed with rate $\beta - 2$ and whenever it hits its past trajectory to obtain $\tilde{Z}_t^{[\beta-2]}$.
3. Draw half-line $l^*$ from $0$ that forms angle $\phi^* \in (0, 2\pi)$ with the initial segment of $Z_t$ distributed according to $|\sin\phi^*|/4$.
4. If $\tilde{Z}_t^{[\beta-2]}$ hits $l^*$, the combined contour $\theta_*$ has $\iota[\theta_*] = 0$, then
   - with probability $\exp(-[\beta+2]\ell(e^*))$ where $e^*$ is $0 \to l^* \cap \tilde{Z}_t^{[\beta-2]}$, set $\theta := \theta_*$;
   - otherwise $\theta := \emptyset$.

   Otherwise, $\theta := \emptyset$.

## Theorem

For $\beta \geq 2$, $\mathcal{A}_D^{[\beta]}$ coincides in law with the union of contours carried as shifted marks by the $M$-marked point process $\mathcal{Y}^{[\beta]}$ in $D$ defined by Papangelou conditional intensity

$$\lambda((x,\theta);\{(x_i,\theta_i)_{i=1}^k\}) = \begin{cases} 4\pi & \text{if } [x+\theta] \cap \cup_{i=1}^k [x_i+\theta_i] = \emptyset, \quad [x+\theta] \subset D \\ 0 & \text{otherwise} \end{cases}$$

with respect to $\mathrm{Lebesgue} \times \Theta_*^{[\beta]}$.

**Interpretation:** $\mathcal{A}_D^{[\beta]}$ coincides in law with the Poisson process on $D$ with intensity $4\pi$ independently marked according to $\Theta_*^{[\beta]}$ on $M$ conditioned on the event that the shifted marks do not intersect and lie fully in $D$.

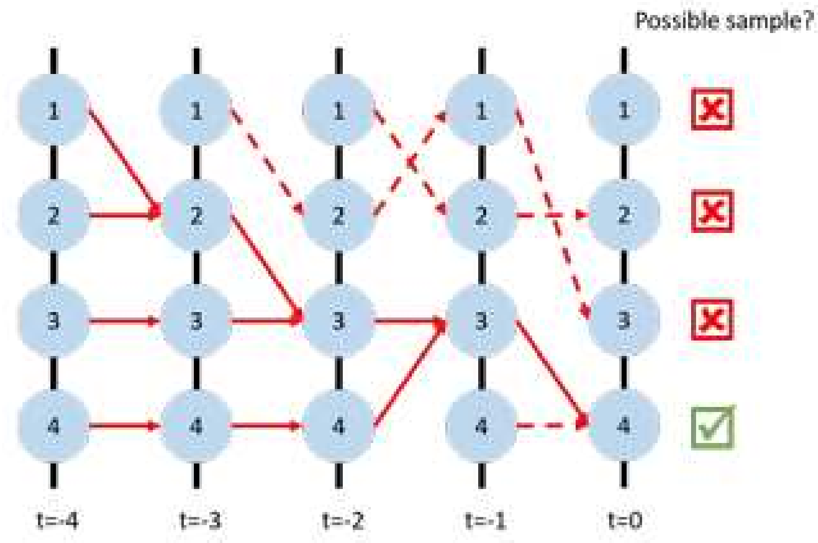
# *Coupling from the past*

(Propp and Wilson, 1995)

Let $S = \{s_1, \ldots, s_M\}$ and $\pi(\cdot)$ be probability mass function on $S$ that is the limit distribution of a Markov chain $X_k$ with update rule

$$X_{k+1} = \phi(X_k, Z_{k+1})$$

for some random variables $Z_0, Z_{-1}, Z_{-2}, \ldots$.
Run Markov chains $(X_k^{(i)})_{k \leq 0}$ starting in each $s_i \in S$ and using **the same** $Z_0, Z_{-1}, Z_{-2}, \ldots$.



# Tricks

For implementation, one needs an efficient way of checking for coalescence. If

- there exists a partial order on the state space with a minimum/maximum;
- the update rule respects the order;

only **two paths** need checking.

In our context, the state space is not finite. Use the **inclusion order** with minimum $\emptyset$ and a **stochastically evolving maximum**:

- $\mathcal{Y}(0)$ a realisation of a Poisson process of rate $4\pi$ in $D$, marked i.i.d. according to $\Theta_*^{[\beta]}$;
- extend $\mathcal{Y}(\cdot)$ backwards by means of a spatial birth-and-death process with birth rate $4\pi dx \Theta_*^{[\beta]}(d\theta)$ and unit death rate.

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# Update rule

- Set $L^{-T}(-T) = \emptyset$ and $U^{-T}(-T) = \mathcal{Y}(-T)$.
- If $\mathcal{Y}(\cdot)$ experiences a backward **birth**, $\mathcal{Y}(t-) = \mathcal{Y}(t) \cup \{(x,\theta)\}$ delete $(x,\theta)$ from $L^{-T}(t-)$ and $U^{-T}(t-)$.
- If $\mathcal{Y}(\cdot)$ experiences a backward **death** $\mathcal{Y}(t-) = \mathcal{Y}(t) \setminus \{(x,\theta)\}$, add $(x,\theta)$ to $L^{-T}(t-)$ iff

$$[x+\theta] \cap \bigcup_{(x_i,\theta_i)\in U^{-T}(t-)} [x_i+\theta_i] = \emptyset, \; [x+\theta] \subseteq D$$

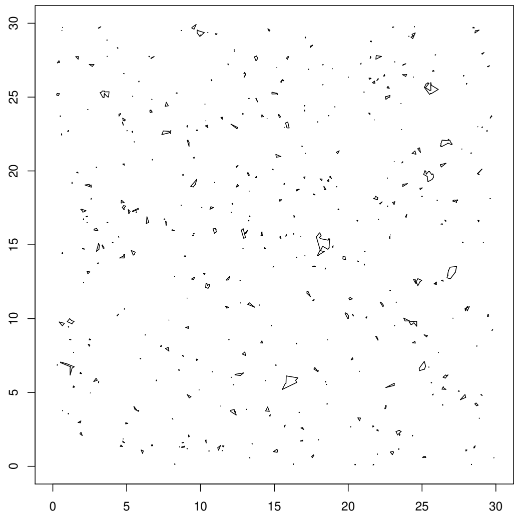
and to $U^{-T}(t-)$ iff

$$[x+\theta] \cap \bigcup_{(x_i,\theta_i)\in L^{-T}(t-)} [x_i+\theta_i] = \emptyset, \; [x+\theta] \subseteq D.$$

(Kendall & Møller, 2000)

# Theorem

$\mathcal{A}_D^{[\beta]}$ coincides in distribution with the output of the coupling from the past algorithm.




$D = [0, 30]^2$, $\beta = 2$

# Conclusion

- Formulated the length-interacting Arak process as a hard object process.
- Derived the mark distribution in terms of a random walk representation.
- Adapted an exact simulation method for fast sampling.

*Thank you for your attention!*