1. Consider a Cox process $X$ on $\mathbb{R}^2$ with driving random measure defined by

$$\Lambda(A) = \int_A L(x) \, dx$$

for bounded Borel sets $A \subseteq \mathbb{R}^2$.

a. Let $L$ be exponentially distributed with rate parameter $\beta > 0$. Compute the first order moment measure of $X$.

b. Again under the assumption that $L$ is exponentially distributed with rate parameter $\beta > 0$, compute the pair correlation function of $X$. Is $X$ second order intensity-reweighted moment stationary? Motivate your answer.

For $N \in \{2, 3, \ldots\}$, let $T = \{1, \ldots, N\}^2 \subset \mathbb{Z}^2$ be a planar cube equipped with the first-order neighbourhood relation $s \sim t$ if and only if $s$ and $t$ are direct horizontal or vertical neighbours. For all $t \in T$, set

$$\beta_t(x_{T \setminus t}) = \alpha + \theta \sum_{s \sim t} x_s$$

and define local characteristics

$$\pi_t(x_t | x_{T \setminus t}) = \beta_t(x_{T \setminus t}) \exp \left[-x_t \beta_t(x_{T \setminus t})\right], \quad x_s \in \mathbb{R}^+, \ s \in T$$

depending on the parameters $\theta \geq 0$ and $\alpha > 0$.

c. Show that the local characteristics define a proper joint probability distribution on $(\mathbb{R}^+)^T$.

d. How would you simulate realisations from the joint probability distribution $\pi_T$ defined by the local characteristics $\pi_t$, $t \in T$? Please give full details.
2. Consider a point process $X$ on the unit square whose distribution is defined by its density

$$
\frac{\beta^{n(x)}|\bigcup_{x \in X} B(x,R)|}{Z(\beta, \gamma, R)}
$$

with respect to a unit rate Poisson process on $[0, 1]^2$. Here, $\beta, \gamma > 0$ are the model parameters, $n(x)$ denotes the cardinality of the point configuration $x$ and $|\bigcup_{x \in X} B(x, R)|$ is the area of the union of closed balls of radius $R > 0$ centred at the points of $x$.

a Calculate the Papangelou conditional intensity of $X$ and interpret your result.

b Is $X$ a pairwise interaction process? Explain your answer.

c Derive the pseudo-likelihood equations for estimating $\beta$ and $\gamma$.

d Assume that $\beta = 5$, $\gamma = 1$ and $R = 0.02$. For each $\epsilon \in [0, 0.5]$, compute the probability that $X$ places exactly four points in the unit square, none of which closer than $\epsilon$ to the boundary of the square. Sketch the graph as a function of $\epsilon$.

3. Consider, for $\alpha \geq 0$ and $\beta > 0$, the semi-variogram

$$
\gamma(t) = \begin{cases} 
\alpha + \beta \left(1 - \frac{\sin(t)}{t}\right), & 0 \neq t \in \mathbb{R} \\
0, & t = 0
\end{cases}
$$

a Sketch the graph of $\gamma(t)$ as a function of $t$. Calculate the nugget, sill and partial sill of this model.

Define, for $\beta > 0$, the function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$
\rho(t, s) = \beta \frac{\sin(t - s)}{t - s}.
$$

b Show that $\rho$ is the covariance function of a stationary Gaussian random field.

[Hint:] You may use that

$$
\int_{-\infty}^{\infty} \cos(wt) \frac{\sin(t)}{t} dt = \frac{\pi}{2} (\text{sgn}(w + 1) - \text{sgn}(w - 1))
$$

where $\text{sgn}(x) = |x|/x$ for $x \neq 0$ with $\text{sgn}(0) = 0$.

c Let $(X_t)_{t \in \mathbb{R}}$ be a Gaussian random field with unknown constant mean $\mu$ and covariance function $\rho$. Suppose that observations are available at $t_1 = \pi/2$ and $t_2 = -\pi/2$ with $(X_{t_1}, X_{t_2}) = (2, 3)$. Predict $X_0$ based on $X_{t_1}$ and $X_{t_2}$.

d What is the mean squared error of your prediction $\hat{X}_0$?