A multi-scale area-interaction model for spatio-temporal point patterns

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The area-interaction model in space

Density

$$p(\mathbf{x}) = \alpha \beta^{n(\mathbf{x})} \gamma^{-A_r(\mathbf{x})}$$

with respect to unit rate Poisson process on bounded $W \subset \mathbb{R}^2$. Here β, is positive, $n(\mathbf{x})$ denotes the cardinality of $\mathbf{x}$ and $A_r(\mathbf{x})$ the area of

$$(\mathbf{x} \oplus B(0, r)) \cap W.$$

- $\gamma = 1$: Poisson process;
- $\gamma > 1$: attraction, cf. Widom–Rowlinson (1970);
- $\gamma < 1$, inhibition, cf. Baddeley and Van Lieshout (1995).

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Multi-scale area-interaction

Consider the **influence function** defined for $u \neq v$ by

$$\kappa(u,v) = \begin{cases} \kappa_j & \text{if } r_{j-1} < ||u-v|| \leq r_j \\ 0 & \text{if } ||u-v|| > r_m \end{cases}$$

for $r_0 = 0 < r_1 < r_2 < \cdots < r_m$ and $1 = \kappa_1 > \kappa_2 > \cdots > \kappa_m > 0$ and set

$$\begin{aligned} p(\mathbf{x}) &= \alpha\beta^{n(\mathbf{x})} \exp\left[-\log\gamma \int_W \max_{x\in\mathbf{x}} \kappa(w,x)dw\right] \\ &= \alpha\beta^{n(\mathbf{x})} \exp\left[-\log\gamma \sum_{j=1}^{m} \kappa_j |\{w \in W : d(w,\mathbf{x}) \in (r_{j-1}, r_j]|\right] \end{aligned}$$

in full analogy to pairwise interaction models based on κ.

Gregori, Van Lieshout and Mateu (2003).

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Product interpretation

Note that

$$
\begin{aligned}
p(\mathbf{x}) &= \alpha\beta^{n(\mathbf{x})} \exp\left[-\log\gamma \sum_{j=1}^{m} \kappa_j \left|((\mathbf{x} \oplus B(0,r_j)) \setminus (\mathbf{x} \oplus B(0,r_{j-1}))) \cap W\right|\right] \\
&= \alpha\beta^{n(\mathbf{x})} \exp\left[-\log\gamma \sum_{j=1}^{m} (\kappa_j - \kappa_{j+1}) |(\mathbf{x} \oplus B(0,r_j)) \cap W|\right] \\
&= \alpha\beta^{n(x)} \prod_{j=1}^{m} (\gamma^{\kappa_j - \kappa_{j+1}})^{-A_{r_j}(\mathbf{x})}
\end{aligned}
$$

under the convention $\kappa_{m+1} = 0$.

Relaxing the constraint that $\kappa_j > \kappa_{j+1}$, inhibition and attraction may be combined, cf. Picard *et al.* (2009) and Ambler and Silverman (2010).

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Space-time area-interaction: Definition

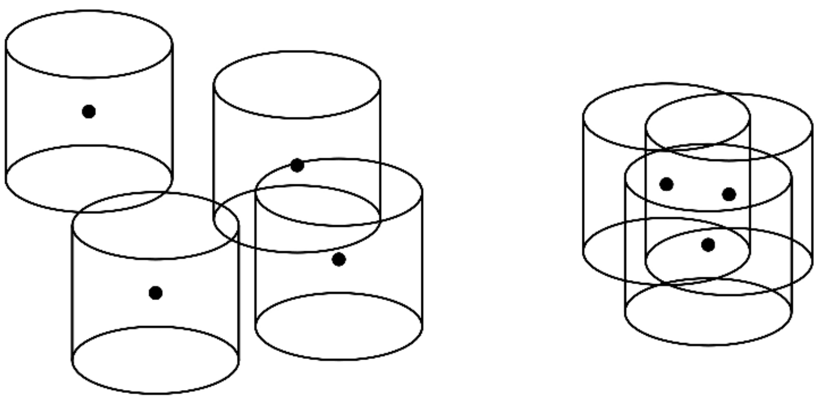
Density

$$p(\mathbf{x}) = \alpha \beta^{n(\mathbf{x})} \gamma^{-\ell(\mathbf{x} \oplus G)}$$

with respect to unit rate Poisson process on bounded subset $W_S \times W_T \subset \mathbb{R}^2 \times \mathbb{R}$. Here $\alpha > 0$ is a normalizing constant,

- ℓ is Lebesgue measure restricted to $W_S \times W_T$,
- $\gamma > 0$ is the interaction parameter, and
- G is a cylinder

$$\mathcal{C}_r^t(0,0) = \{(y,s) \in W_S \times W_T : ||y|| \le r, |s| \le t\}.$$



Multi-scale space-time area-interaction

Consider the **influence function** defined for $u \neq v$ by

$$\kappa((u_S, u_T), (v_S, v_T)) = \begin{cases} \kappa_j & \text{if } (u_S - v_S, u_T - v_T) \in G_j \setminus G_{j-1} \\ 0 & \text{if } (u_S - v_S, u_T - v_T) \notin G_m \end{cases}$$

where, for $r_0 < r_1 < r_2 < \cdots < r_m$, $t_0 < t_1 < t_2 < \cdots < t_m$, $G_1 \subset \cdots \subset G_m$ are nested cylinders $C_{r_j}^{t_j}(0,0)$, $1 = \kappa_1 > \kappa_2 > \cdots > \kappa_m$ and set

$$\begin{aligned} p(\mathbf{x}) &= \alpha \beta^{n(\mathbf{x})} \exp\left[-\log\gamma \int_{W_S \times W_T} \max_{(x,t) \in \mathbf{x}} \kappa((w_S, w_T), (x,t)) dw_S dw_T \right] \\ &= \alpha \beta^{n(\mathbf{x})} \exp\left[-\log\gamma \sum_{j=1}^{m} \kappa_j \ell((\mathbf{x} \oplus G_j) \setminus (\mathbf{x} \oplus G_{j-1})) \right]. \end{aligned}$$

Product interpretation and Markov property

Since

$$p(\mathbf{x}) = \alpha \beta^{n(\mathbf{x})} \prod_{j=1}^{m} \gamma_j^{-\ell(\mathbf{x} \oplus G_j)}$$

for

$$\log \gamma_j = (\kappa_j - \kappa_{j+1}) \log \gamma$$

under the convention $\kappa_{m+1} = 0$, the log **conditional intensity** is

$$\begin{aligned} \log \lambda((y,s);\mathbf{x}) &= \log \left(\frac{p(\mathbf{x} \cup \{(y,s)\})}{p(\mathbf{x})} \right) \\ &= \log \beta - \sum_{j=1}^{m} \log \gamma_j \ell \left(((y,s) \oplus G_j) \setminus (\mathbf{x} \oplus G_j) \right) \end{aligned}$$

and hence p is Markov at range $2 \max\{r_m, t_m\}$.

Note: generalise to any γ_j by relaxing the constraint $\kappa_j > \kappa_{j+1}$.

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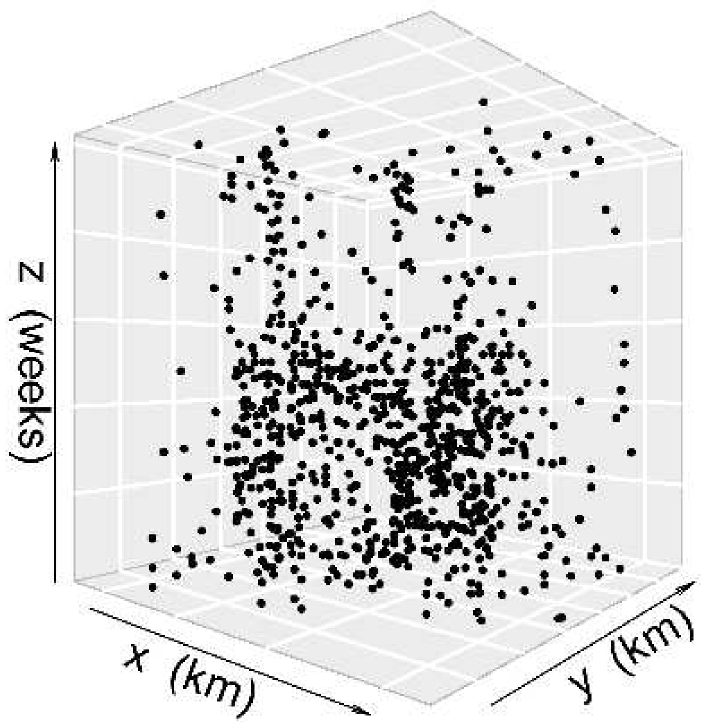
Varicella

- transmitted by direct contact with the rash or by inhalation of aerosolised droplets from respiratory tract secretions of patients;
- mostly mild in childhood, severe in adults;
- may be fatal in neonates and immuno-compromised people;
- 10 to 21 days incubation period;
- itchy, vesicular rash, fever and malaise;
- it takes 7 to 10 days for vesicles to dry out.

Data

921 cases in 16 districts in Valencia, Spain, during 2013.

- Bounding region $W_S = [0,9] \times [0,9]\,\mathrm{km}^2$.
- Bounding time region $W_T = [0,52]$ weeks.



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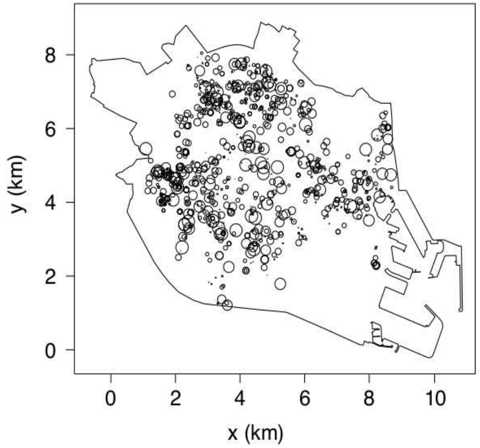
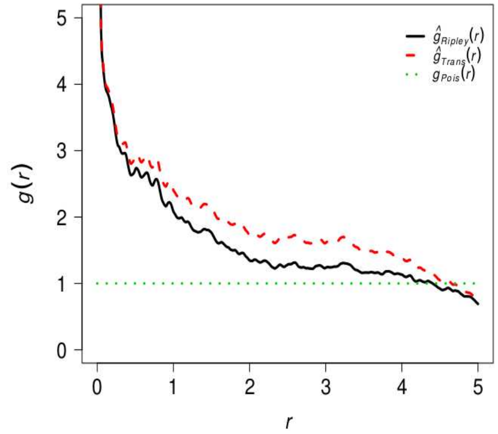
Exploratory analysis – Spatial component





Left: projection in space. To get rid of duplicate locations, add jitter in $(0, 0.01)$ to each coordinate.
Right: estimated pair correlation function assuming stationarity and isotropy, Epanechnikov kernel

$$\kappa_\epsilon(t) = \frac{3}{4\epsilon}\left(1 - \frac{t^2}{\epsilon^2}\right), \quad -\epsilon \leq t \leq \epsilon,$$

with bandwidth according to Stoyan's rule of thumb $0.15(5\hat{\lambda})^{-1/2}$.
Conclusion: $r_m \approx 2$.

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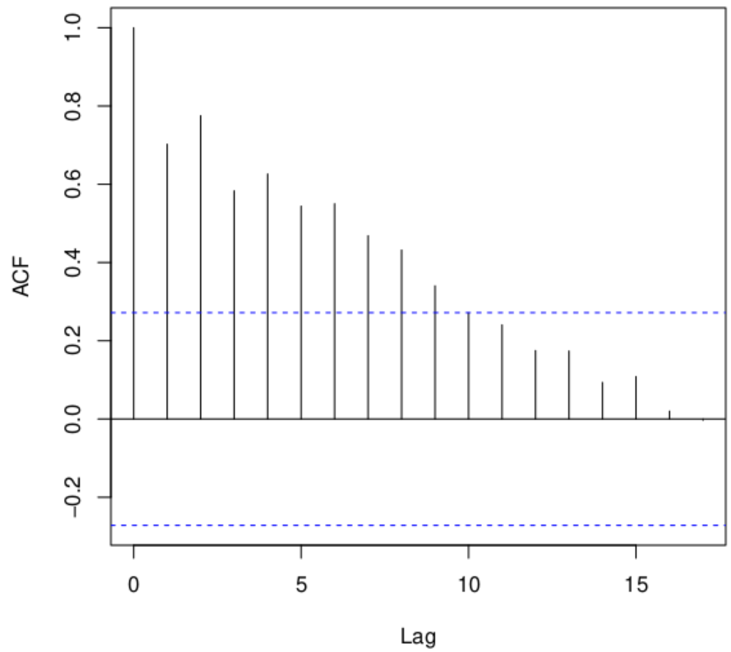
Exploratory analysis – Temporal component



Projection in time and estimated auto-correlation function assuming stationarity.

Conclusion: $t_m \approx 7.5$.

Modelling – Covariate information

Idea: add a first order interaction function based on explanatory variables.

Visual inspection suggests a **separable** function

$$\lambda(x,t) = \beta\lambda(x)Z(t), x \in [0,9]^2, t \in \{0,\ldots,51\}$$

is reasonable, where $\beta > 0$, $\lambda(x)$ is a non-parametric estimate of the population density, and $Z(t)$ a fitted harmonic regression

$$Z(t) = c_0 + \sum_{j=1}^{3}(c_j \cos(2\pi jt/52) + d_j \sin(2\pi jt/52)) + c(a+bt)$$

rescaled by 100.

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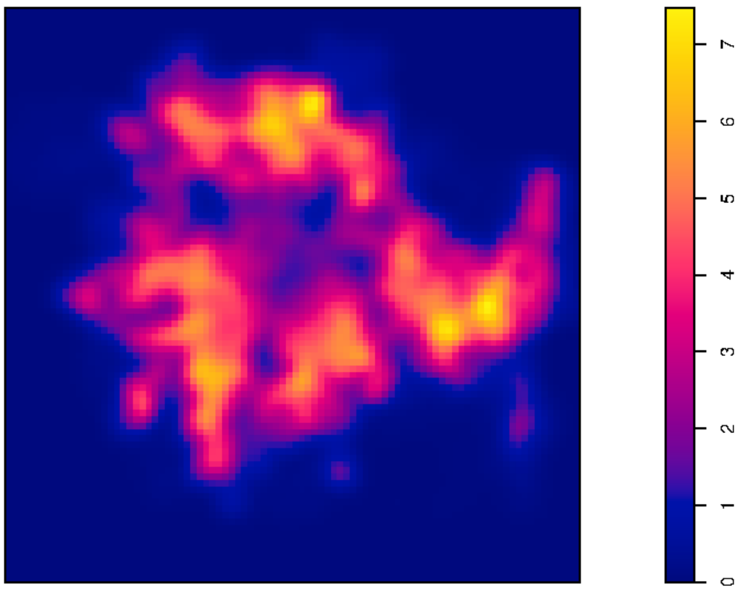
Results and details

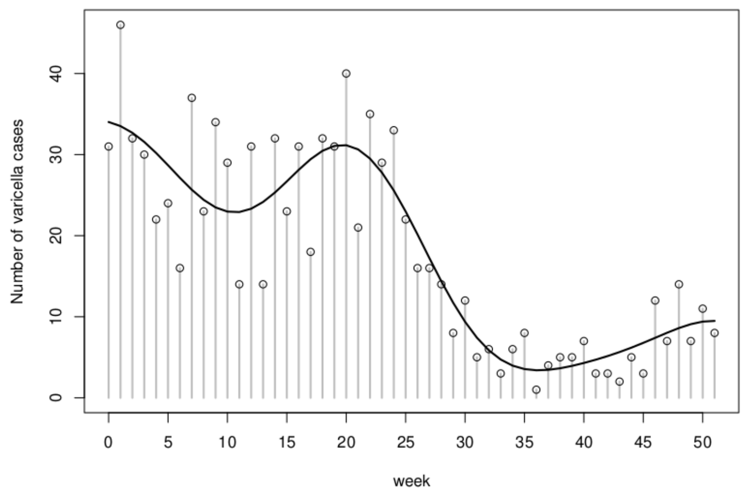
$\lambda(x)$:

Known: number N_i of inhabitants in 559 sections in the 16 districts. Take N_i uniformly distributed points in each section, apply a Gaussian kernel smoother with $\sigma = 0.15$ and global edge correction, and scale by $1,000$.

$Z(t)$:

Estimate $c_0, \ldots, c_3$, $d_1, \ldots d_3$, c, a and b by maximum likelihood in the Gaussian family.



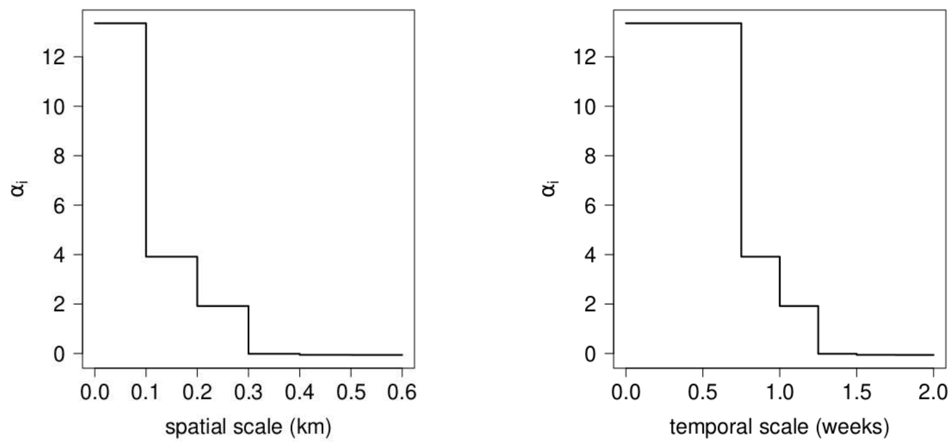


Inference

Starting from $r_m = 2$ and $t_m = 7.5$, we zoom in by discarding ranges where the estimated parameters κ oscillate around zero to arrive at the following **model:**

$$p(\mathbf{x}) \propto \beta^{n(\mathbf{x})} \prod_{(x,t)\in\mathbf{x}} \lambda(x,t) \exp\left[-\sum_{j=1}^{6} \kappa_j \ell((\mathbf{x}\oplus G_j) \setminus (\mathbf{x}\oplus G_{j-1}))\right]$$

for $r = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$ and $t = (0.75, 1.0, 1.25, 1.5, 1.75, 2.0)$.



Estimating the parameters

Nguyen–Zessin–Georgii equation

$$\mathbb{E}\left[\sum_{x \in X \cap W_S \times W_t} h(x, X \setminus \{x\})\right] = \mathbb{E}\left[\int_{W_S}\int_{W_T} h(x, X)\lambda_\theta(x; X)dx\right].$$

The **Takacs–Fiksel** idea is to choose convenient functions h, estimate both sides, equate and solve for θ.

Pseudo-likelihood: (Besag, 1977; Jensen and Møller, 1991)

$$h(x, X) = \frac{\partial}{\partial\theta}\log\lambda_\theta(x; X)$$

Logistic regression: (Baddeley et al. 2014)

$$h(x, X) = \frac{\partial}{\partial\theta}\log\left[\frac{\lambda_\theta(x; X)}{\lambda_\theta(x; X) + \rho(x)}\right]$$

with $\rho(x) = \lambda(x)Z(t)/25$.

Validation

- 99 realisations from the fitted model in `MPPLIB` using Metropolis–Hastings.
- Monte Carlo envelopes for estimates of $L_{\mathrm{inhom}}(r, t = 3r)$ (Gabriel and Diggle, 2009) using `stpp`.