On the existence of 0/1 polytopes with high semidefinite extension complexity

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Joint Work with...

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- Jop Briët (CWI)

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Alternative measure of complexity independent from P vs NP.

Lower bounds on the size of *Linear Programs*:

1. Any symmetric LP that captures the TSP or Matching polytope must have size $2^{\Omega(n)}$ [**Yannakakis** '91]
2. There exists a convex hull of $0/1$ points that cannot be captured by an LP of size less than $2^{\Omega(n)}$ [**Rothvoss** '11]
3. Any LP that captures the TSP polytope must have size $2^{\Omega(n^{1/2})}$ [**Fiorini, Massar, Pokutta, Tiwary, de Wolf** '12]
4. Any LP that ρ-approximates the Correlation polytope must have size $2^{\Omega(n/\rho)}$ [**Braun, Fiornini, Pokutta, Steurer** '12, **Braverman, Moitra** '13, **Pokutta, Braun** '13]
5. Any LP of relaxation of size n^r for the Correlation polytope has integrality gap at least as large as $O(r)$ levels of Sherali-Adams [**Chan, Lee, Raghavendra, Steurer** '13]

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What about *Semidefinite Programs*?

Theorem ([Briët, D., Pokutta '13])

There exists a convex hull of $0/1$ points that cannot be captured by an SDP of size less than $2^{\Omega(n)}$.

Extend framework of Rothvoss to SDP setting.
Baby step towards understanding lower bounds for SDPs.

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Polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with m facets.

Question

Can we reduce the number of inequalities needed to define P by adding auxilliary variables?

Definition (Linear Extension)

$Q = \{(z, y) : Cz + Dy = d, y \geq 0, y \in \mathbb{R}^r, z \in \mathbb{R}^l\}$, is a linear extension of P of size r if $\exists\ \pi : \mathbb{R}^{l+r} \to \mathbb{R}^n$ such that

$$P = \pi(Q).$$

Definition (Linear Extension Complexity)

$\mathrm{xc}(P) :=$ minimum size of any linear extension of P.

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Extension Complexity.

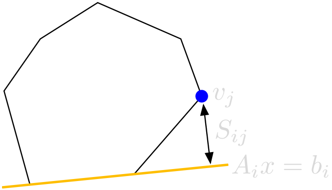
Some known results (constructions & lower bounds):

- $\text{xc}($regular n-gon$) = \Theta(\log n)$ [Ben-Tal, Nemirovski'01]
- $\text{xc}($generic n-gon$) = \Omega(\sqrt{n})$ [Fironi, Rothvoss, Tiwary'11]
- $\text{xc}(n$-permutahedron$) = \Theta(n \log n)$ [Goemans'09]
- $\text{xc}($spanning tree polytope of $K_n) = O(n^3)$ [Kipp-Martin'87]
- $\text{xc}($spanning tree polytope of planar graph $G) = \Theta(n)$ [Williams'01]
- $\text{xc}($stable set polytope of perfect graph $G) = n^{O(\log n)}$ [Yannakakis'91]
- ...

Slack Matrices.

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $\mathcal{V} = \{v_1, \ldots, v_N\} \subseteq \mathbb{R}^n$ s.t.

$$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\} = \text{conv}(\mathcal{V})$$



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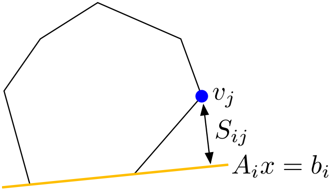

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Definition

A rank-r nonnegative factorization of $S \in \mathbb{R}_{+}^{m \times n}$ is

$$S = UV \quad \text{where} \quad U \in \mathbb{R}_{+}^{m \times r} \quad \text{and} \quad V \in \mathbb{R}_{+}^{r \times n}$$

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Nonnegative Factorizations and Extensions.

Definition (Nonnegative Rank)

$$\mathrm{rk}_+(S) := \min\{r \mid \exists \text{ rank-}r \text{ nonnegative factorization of } S\}$$

Theorem (Factorization Theorem [Yannakakis'91])

For every slack matrix S of P:

$$\mathrm{xc}(P) = \mathrm{rk}_+(S)$$

PSD Matrices.

Definition (PSD matrix)

A matrix $U \in \mathbb{R}^{r \times r}$ is PSD if U is symmetric and

$$x^\mathsf{T} U x \geq 0 \quad \forall x \in \mathbb{R}^r.$$

Let $\mathbb{S}_+^r$ denote the set of $r \times r$ PSD matrices.

Definition (Spectral Decomposition)

U is $r \times r$ PSD iff U admits a *Spectral Decomposition*

$$U = \sum_{i=1}^{r} \lambda_i u_i u_i^\mathsf{T},$$

$\lambda_1, \ldots, \lambda_r \geq 0$, $u_1, \ldots, u_r$ an orthonormal basis.

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Definition (Operator norm)

For a matrix $T \in \mathbb{R}^{r \times r}$ the operator norm of T is

$$\|T\|_{\text{op}} = \max_{\|x\|_2 = 1} \|Tx\|_2$$

For a PSD matrix $U \in \mathbb{R}^{r \times r}$

$$\|U\|_{\text{op}} = \max_{\|x\|_2 = 1} x^{\mathsf{T}} U x = \text{ largest eigenvalue of } U.$$

PSD Matrices.

Definition (Trace)

For a matrix $T \in \mathbb{R}^{r \times r}$, we define $\mathrm{Tr}[T] = \sum_{i=1}^{r} T_{ii}$.

Remark (Trace Inner Product)

For $A, B \in \mathbb{R}^{r \times r}$ symmetric, $\mathrm{Tr}[AB] = \sum_{i,j \in [r]} A_{ij} B_{ij}$.

Proposition

For PSD matrices $U, V \in \mathbb{S}_+^r$,

$$\mathrm{Tr}[UV] = \sum_{i,j \in [r]} \lambda_i \gamma_j \langle u_i, v_j \rangle^2 \geq 0 \ ,$$

where $U = \sum_{i=1}^{r} \lambda_i u_i u_i^\mathsf{T}$ and $V = \sum_{j=1}^{r} \gamma_j v_j v_j^\mathsf{T}$ are the respective spectral decompositions.

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Theorem (Factorization Theorem [Gouviea, Thomas, Parillo '11, Fiorini, Massar, Pokutta, Tiwary, de Wolf '12])

For every slack matrix S of P:

$$\mathrm{xc}_{\mathrm{psd}}(P) = \mathrm{rk}_{\mathrm{psd}}(S)$$

Rothvoss's Counting Argument

Goal:
Show existence of $X \subseteq \{0,1\}^n$ with $\mathrm{xc}(\mathrm{conv}(X)) = 2^{\Omega(n)}$.

Let $R = \max_{X \subseteq \{0,1\}^n} \mathrm{xc}(\mathrm{conv}(X))$.

High level:

1. "Discretize" an optimal a linear EF for $\mathrm{conv}(X)$ to *compress the description* of X.
2. Show that the number of discretized linear EFs of size R is bounded by $2^{\mathrm{poly}(R,n)}$.

Conclusion: 2^{2^n} subsets of $\{0,1\}^n$ means that $R \geq 2^{\Omega(n)}$.

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Examine linear EF $Q = \{(x, y) : Ax + Uy = b, y \geq 0\}$.

Let (A_S, U_S), $S \subseteq [m]$, be any maximal set of linearly independent set of rows of (A, U).

Note that $Q = \{(x, y) : A_S x + U_S y = b_S, y \geq 0\}$.

Can represent $\mathrm{conv}(X)$ using only at most $(n + r) \times (n + r + 1)$ numbers!

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$$\bar{Q} = \{(x,y) : \|A_S x + \bar{U}_S y - b_S\|_\infty \leq 1/\mathrm{poly}(n,r), \quad y \geq 0, \|y\|_\infty \leq \sqrt{\Delta}\}.$$

High Level Fix:

1. Rescale factorization $S = UV$ so that max entry $U, V \leq \sqrt{\Delta}$.
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Question: What changes for PSD factorizations?

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Theorem ([Briët, D., Pokutta])

Let $S \in \mathbb{R}_+^{m \times N}$ *with* $\mathrm{rk}_{\mathrm{psd}}(S) = r$ *and* $\|S\|_\infty = \Delta$.
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Rescaling Nonnegative Factorizations

Lemma

Let $S \in \mathbb{R}_+^{m \times N}$ *with* $\mathrm{rk}_+(S) = r$ *and* $\|S\|_\infty = \Delta$.
Then $\exists$ *rank* r *nonnegative factorization* $S = UV$ *such that*

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Rescaling Nonnegative Factorizations

Proof: $U = \begin{pmatrix} u_1 & \dots & u_r \end{pmatrix}$, $V = \begin{pmatrix} v_1^\mathsf{T} \\ \vdots \\ v_r^\mathsf{T} \end{pmatrix}$

$S = \sum_{i=1}^r u_i v_i^\mathsf{T}$.

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Choose $\lambda_1, \ldots, \lambda_r > 0$ such that $\|\lambda_i u_i\|_\infty = \|1/\lambda_i v_i\|_\infty$.

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Note that $U'V' = \sum_{i=1}^r (\lambda_i u_i)(1/\lambda_i v_i^\top) = \sum_{i=1}^r u_i v_i^\top = S$.

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Comparison with Nonnegative Factorizations:

- Nonnegative setting: can rescale entries of non-negative vector independently and maintain non-negativity.
- PSD setting: entries can be NEGATIVE.
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Open problems and future work.

1. Is the rescaling theorem tight? Do we need dependance on r?
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Thank you!