On the Shadow Simplex Method for Curved Polyhedra

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Outline

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1. Introduction
   - Linear Programming and its Applications
   - The Simplex Method
   - Results
2. The Shadow Simplex Method
   - The Normal Fan
   - Primal and Dual Perspectives
3. Well-conditioned Polytopes
   - τ-wide Polyhedra
   - δ-distance Property
4. Diameter and Optimization
   - 3-step Shadow Simplex Path
   - Bounding Surface Area Measures of the Normal Fan
   - Finding an Optimal Facet

Linear and Integer Programming

- Linear Programming (LP): linear constraints & linear objective with continuous variables.

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- Amazingly versatile modeling language.
- Generally provides a "relaxed" view of a desired optimization problem, but can be solved in polynomial time via interior point (and many other) methods!
- Will focus on one of the most used classes of algorithms for LP: the Simplex Method (not a polytime algorithm!).

Linear and Integer Programming

- Mixed Integer Programming (MIP): models both continuous and discrete choices.

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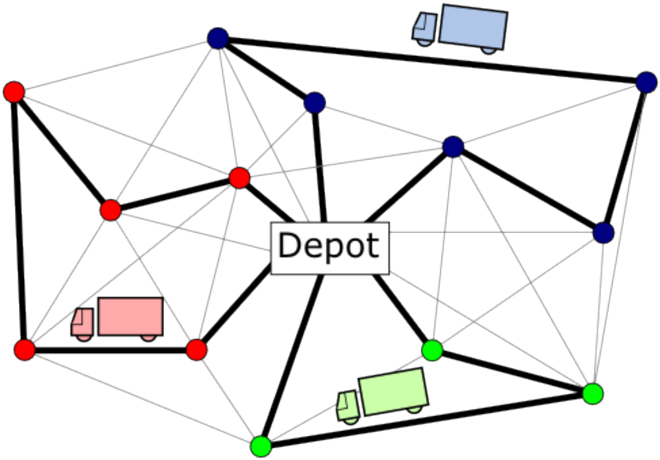
- Many sophisticated software packages exist for these models (CPLEX, Gurobi, etc.). MIP solving is now considered a *mature and practical* technology.

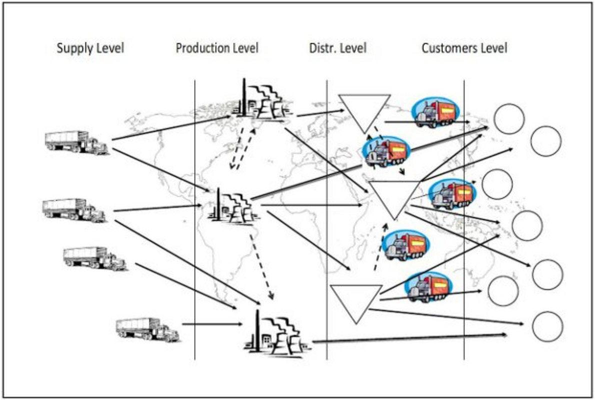
Sample Applications

- Routing delivery / pickup trucks for customers.



Sample Applications

- Optimizing supply chain logistics.



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- Solve the LP.
- Add extra constraints to tighten the LP or “guess” the values of some of the integer variables. Repeat.
- Need to solve a lot of LPs quickly.

Linear Programming via the Simplex Method

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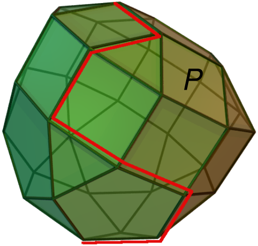
Simplex Method: move from vertex to vertex along the graph of P until the optimal solution is found.



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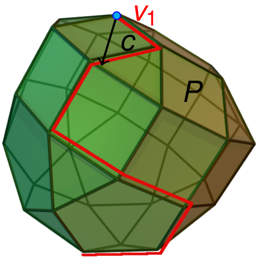

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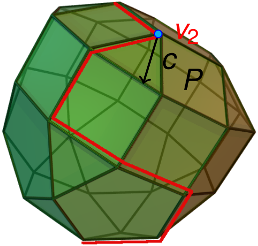

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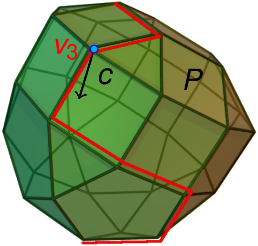

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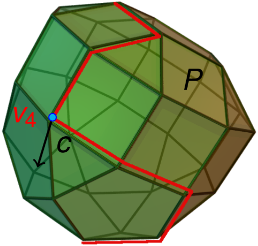

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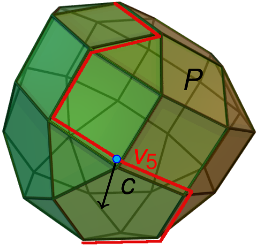

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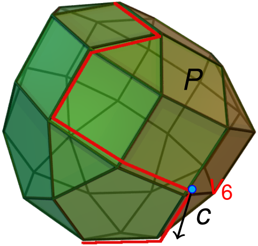

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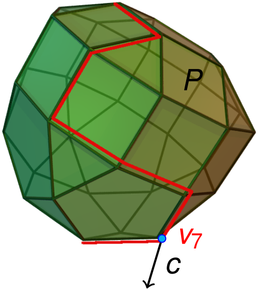

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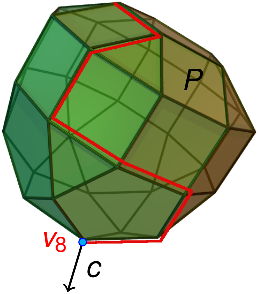

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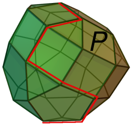

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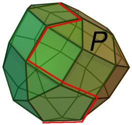

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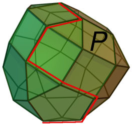

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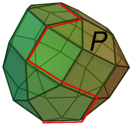

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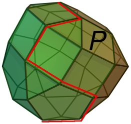

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Why simplex?

- Simplex pivots implementable using “simple” linear algebra.
- Vertex solutions are often “nice” (e.g. sparse, easy to interpret).
- Terminates with combinatorial description of an optimal solution.
- “Easy” to reoptimize when adding an extra variable (dual to adding a constraint).

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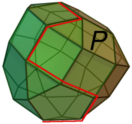

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No known pivot rule is proven to converge in polynomial time!!!

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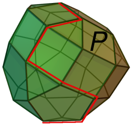

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Simplex lower bounds:

- Klee-Minty (1972): designed “deformed cubes”, providing worst case examples for many pivot rules.
- Friedmann et al. (2011): systematically designed bad examples using Markov decision processes.
- In these examples, the pivot rule is tricked into taking an (sub)exponentially long path, even though short paths exists.

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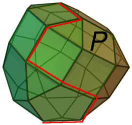

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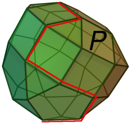
Linear Programming and the Hirsch Conjecture

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$
$$A \in \mathbb{R}^{m \times n}$$



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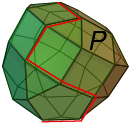

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Besides the computational efficiency of the simplex method, an even more basic question is not understood:

Question

*How can we bound the length of paths on the graph of P? I.e. how to bound the **diameter** of P?*

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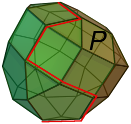

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Conjecture (Polynomial Hirsch Conjecture)

The diameter of P is bounded by a polynomial in the dimension n and number of constraints m.

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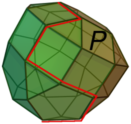

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Diameter lower bounds:

- Santos (2010), Matschke-Santos-Weibel (2012):
  Disproved original Hirsch conjecture bound of $m - n$,
  exhibit polytopes with diameter $(1 + \varepsilon)m$ (for some small $\varepsilon > 0$).

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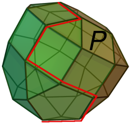

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Diameter upper bounds:

- Barnette, Larman (1974): $\frac{1}{3}2^{n-2}(m - n + \frac{5}{2})$.
- Kalai, Kleitman (1992), Todd (2014): $(m-n)^{\log n}$.

Special Cases

$$P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}, \quad A \in \mathbb{Q}^{m \times n}$$

Upper bounds for combinatorial classes:

- $0/1$-polytopes: $m - n$ (Naddef 1989)
- flow polytopes: quadratic (Orlin 1997)
- transportation polytopes: linear (Brightwell, v.d. Heuvel and Stougie 2006)
- polars of flag polytopes: $m - n$ (Adripasito, Benedetti 2014)

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Upper bounds for *well-conditioned* constraint matrices:

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  - Use volume expansion on the normal fan (non-constructive!).

Simplex Algorithms

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Subdeterminants of A bounded by Δ.

Question

Can the diameter bound of Bonifas et al bounds be made constructive? How fast can we solve LP in this setting?

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- All the above results hold with respect to more general conditions on P (more details later).

A Faster Shadow Simplex Method

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Theorem (D., Hähnle 2014+)

- *Diameter is bounded by* $O(n^3 \Delta^2 \ln(n\Delta))$.

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- *Can compute an initial feasible vertex using* $O(n^5 \Delta^2 \ln(n\Delta))$ *pivots on expectation.*

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Subdeterminants of A bounded by Δ.

Theorem (D., Hähnle 2014+)

- *Diameter is bounded by* $O(n^3 \Delta^2 \ln(n\Delta))$.
- *Given an initial vertex and objective, can compute optimal vertex using at most* $O(n^4 \Delta^2 \ln(n\Delta))$ *pivots on expectation.*
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Based on a new analysis and variant of the **shadow simplex method**. Inspired by path finding algorithm over the Voronoi graph of a lattice by Bonifas, D. (2014) used for solving the Closest Vector Problem.

Navigation over the Voronoi Graph

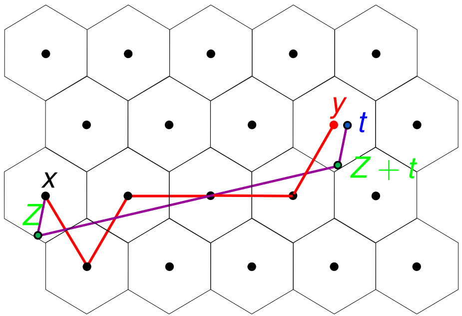

Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.

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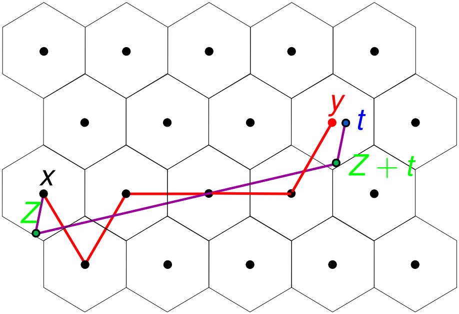


Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.
- Solving CVP can be reduced to efficiently navigating over the Voronoi cell (Som.,Fed.,Shal. 09; Mic.,Voulg. 10-13).

Navigation over the Voronoi Graph

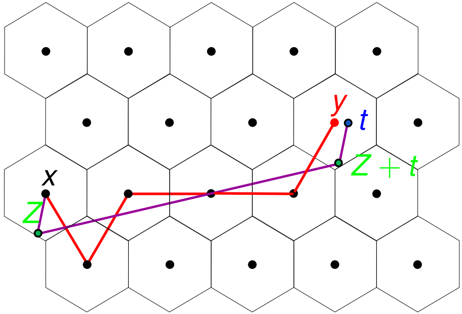


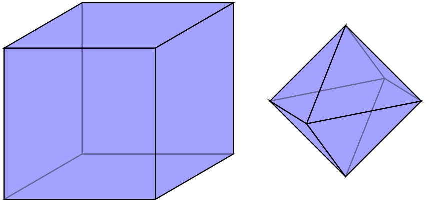
Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.
- Can move between "nearby" lattice points using a polynomial number of steps (Bonifas, D. 14).

Outline

The Polar

- Polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $0 \in int(P)$
- Polar: $P^\star = \{y \in \mathbb{R}^n \ : \ y^T x \leq 1 \, \forall x \in P\}$



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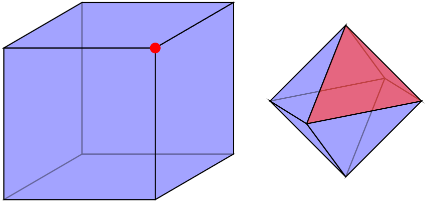

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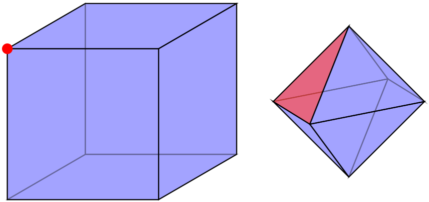

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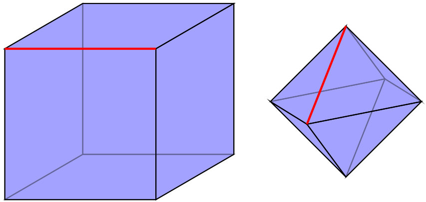

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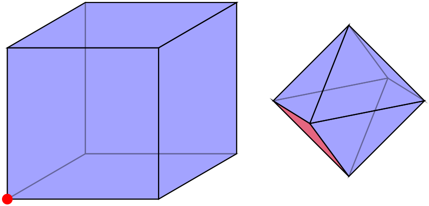

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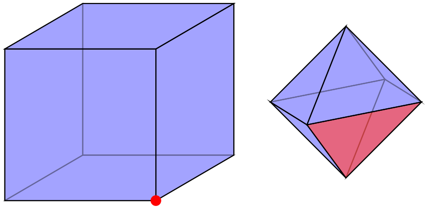

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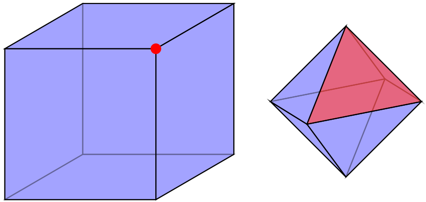

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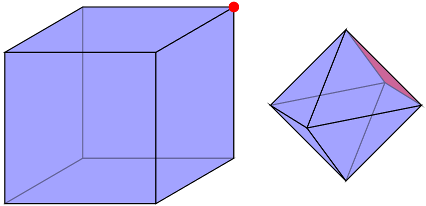

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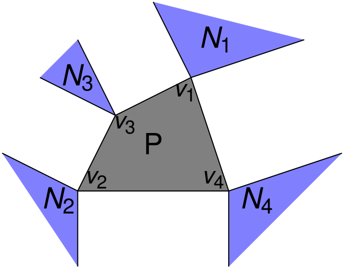
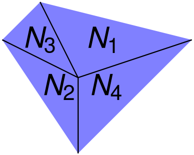
- Same combinatorics as the polar, but expressed using cones.





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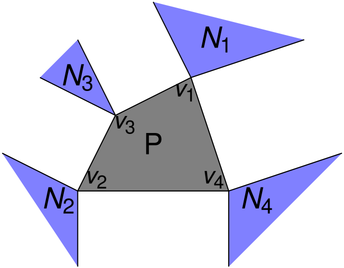

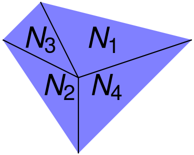

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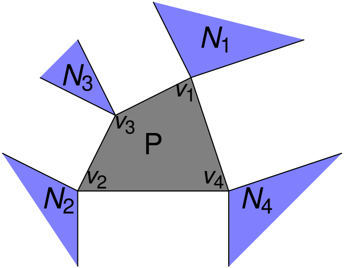

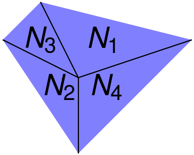

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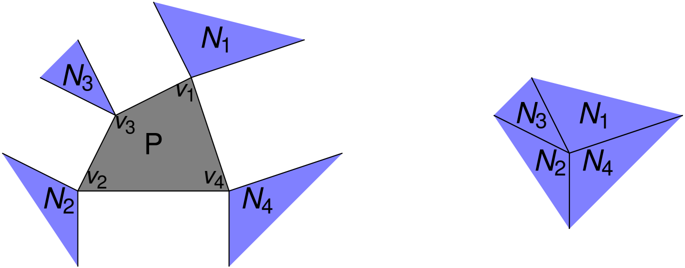
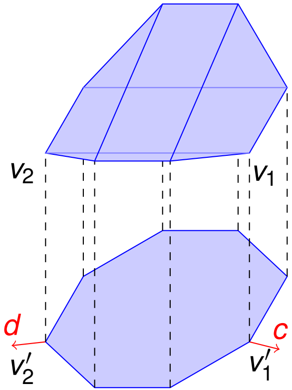
- Normal fan: Set of all normal cones.



The Shadow Simplex Method



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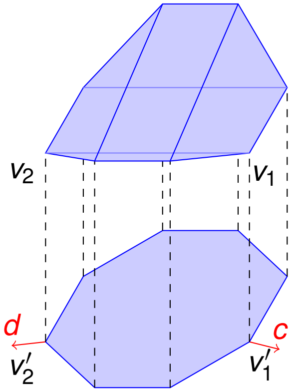

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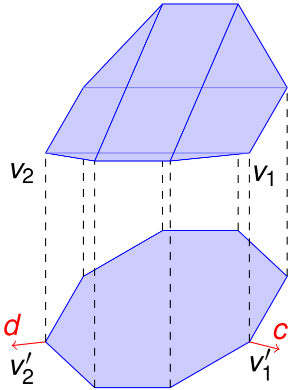

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  - Project P to span of c and d.
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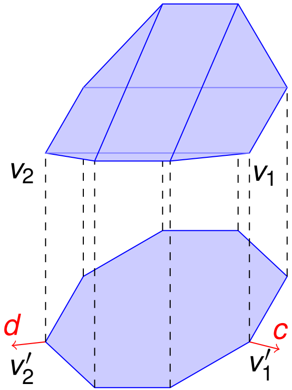

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- "Dual" interpretation
  - Trace segment $[c, d]$ through normal fan.
  - Pivot step corresponds to crossing facet of a normal cone.

Size of the shadow: randomness to the rescue

Question

When can we bound the number of edges in the shadow?

- In general, the shadow can be exponentially large.

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- Brunsch-Röglin (2013): the shadow is small in expectation for "well-conditioned" polytopes when c, d are randomly chosen from the normal cones of two vertices.

Shadow Simplex: Dual Perspective

- Move from v_1 to v_2 by following $[c, d]$ through the normal fan.
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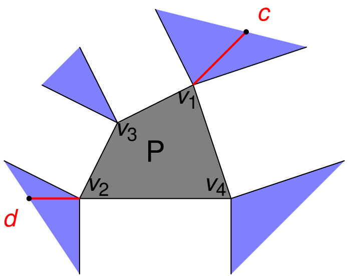

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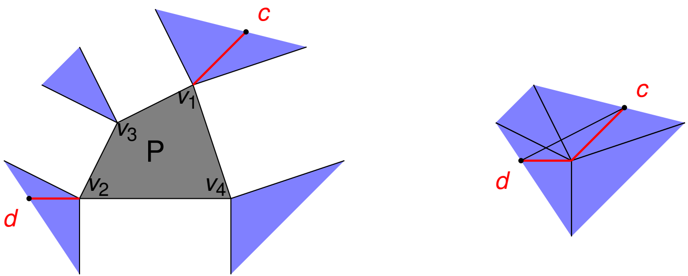

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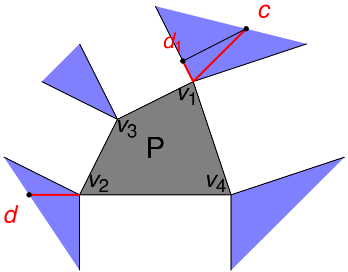

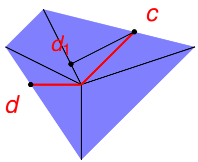

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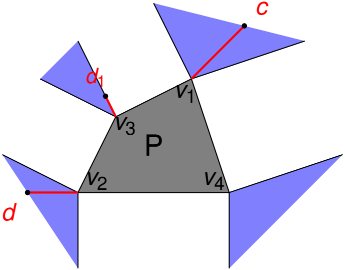

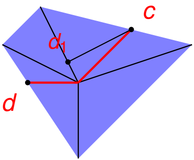

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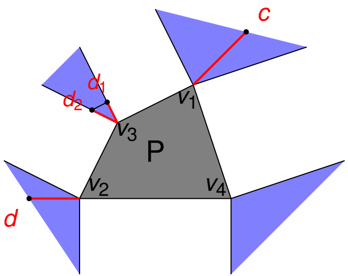

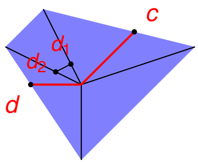

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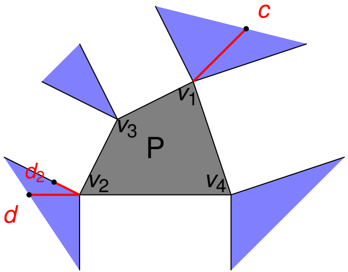

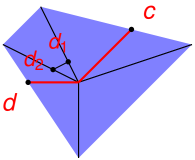

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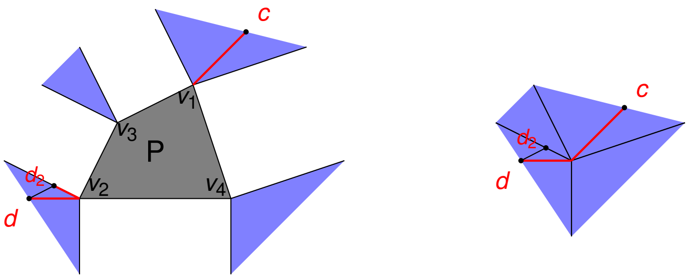

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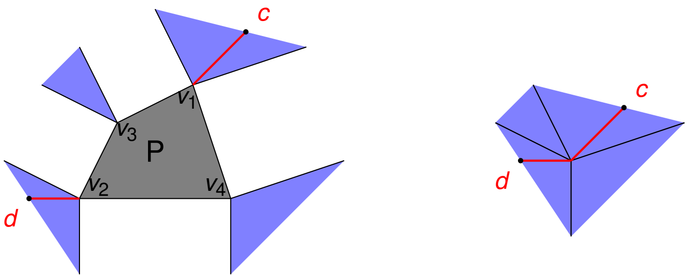


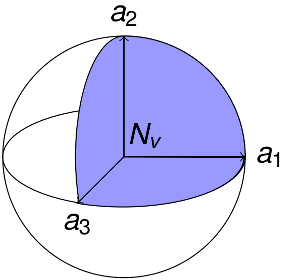
Question

How can we bound the number of intersections with the normal fan?

Outline

Polyhedra with τ-wide Normal Fan

- Vertex normal cone N_v is τ-wide: contains a ball of radius τ centered on the unit sphere.



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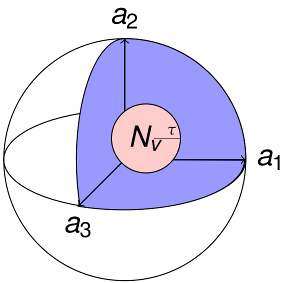

Polyhedra with τ-wide Normal Fan

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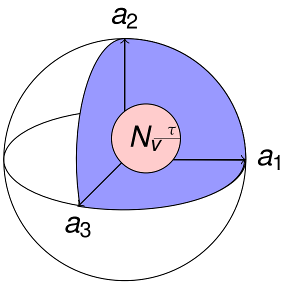
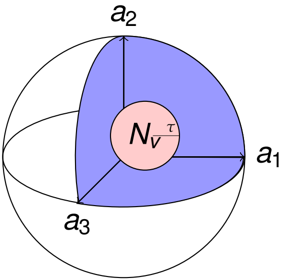
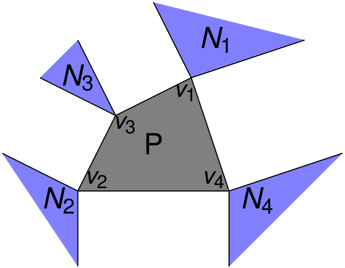
- P is τ-wide if all its vertex normal cones are τ-wide.



Polyhedra with τ-wide Normal Fan

- Vertex normal cone N_v is τ-wide: contains a ball of radius τ centered on the unit sphere.
- Angles at any vertex are less than $\pi - 2\tau$. "Discrete measure" of curvature.





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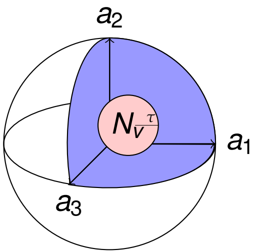

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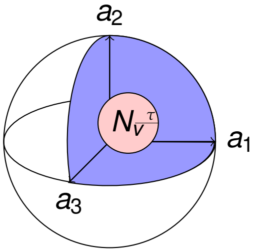
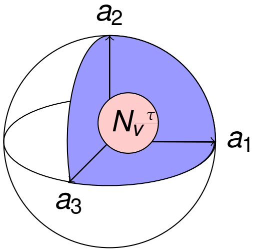


Lemma

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$, $A \in \mathbb{Z}^{m \times n}$, subdeterminants bounded by Δ. Then P is τ-wide for $\tau = 1/(n\Delta)^2$.

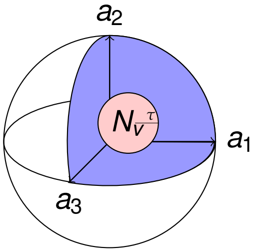
Polyhedra with τ-wide Normal Fan



Theorem (D.-Hähnle 2014+)

If P an n-dimensional polyhedron with a τ-wide normal fan, then diameter of P is $O(n/\tau \ln(1/\tau))$.

Polyhedra with τ-wide Normal Fan



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Furthermore, paths are constructed using shadow simplex method.

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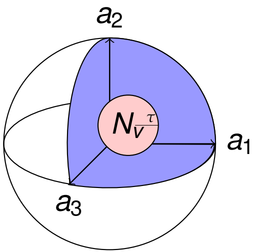

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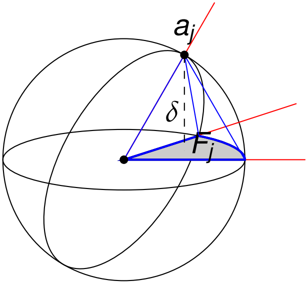
Remark: Perfect matching polytope on a graph $G = (V, E)$ is $1/(3\sqrt{|E|})$-wide.

The δ-distance Property

- $N_v = \mathrm{cone}(a_1, \ldots, a_n)$, a_i's scaled to be unit length.
- Take a_j and opposite facet F_j.



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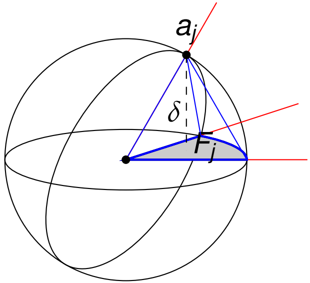

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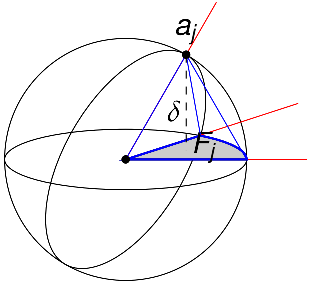

- P has the (local) δ-distance property if every (feasible) basis has the δ-distance property.

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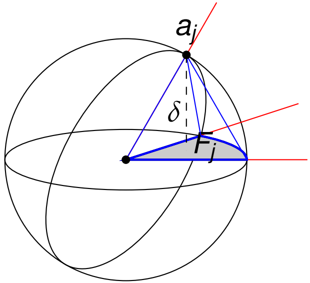

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Polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

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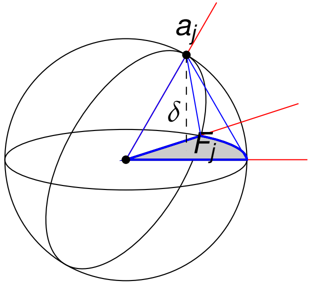

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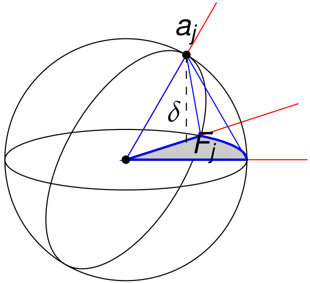

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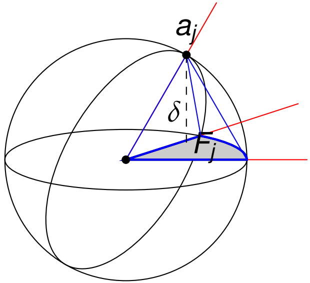

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Resolves question of Vempala and Eisenbrand (2014) regarding sufficiency of *local* δ-distance property for optimization.

Outline

3-step Shadow Simplex Path

To *bound the diameter*, we will exhibit a short shadow simplex path between any two vertices.

- Let v, w be vertices of P.

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Question

How long is this path?

Crossing Bounds

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Next up: Diameter and Phase (b) crossing bound.

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Vertices v, w of P optimized by c, d respectively.

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How to choose c and d?

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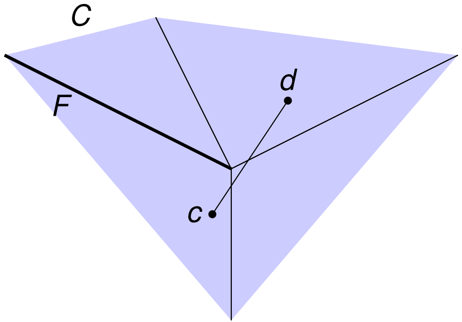
Phase (b): Bound Facet Crossing Probability



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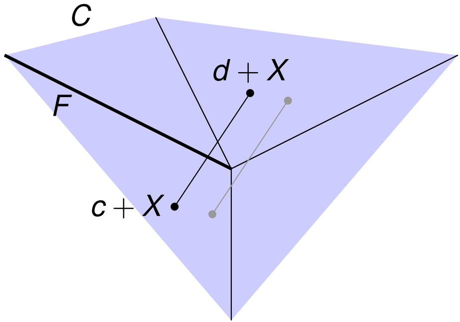

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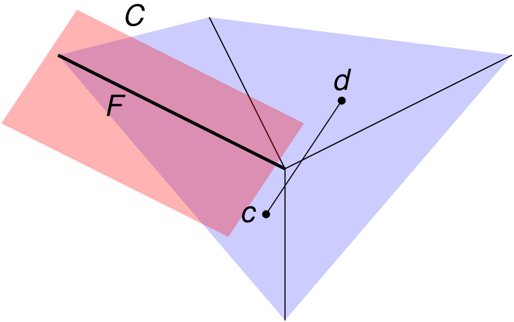

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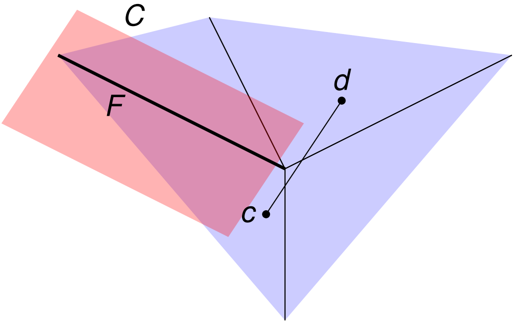

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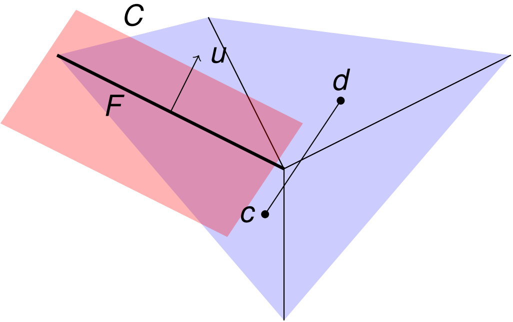

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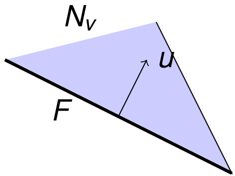
Bounding the Surface Measure using τ



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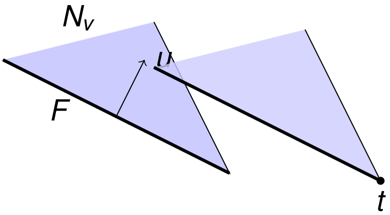

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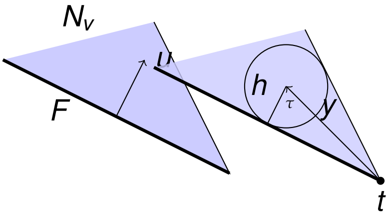

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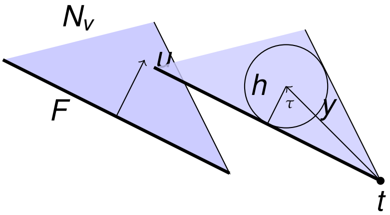

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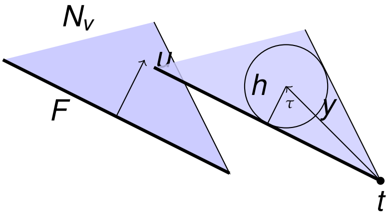

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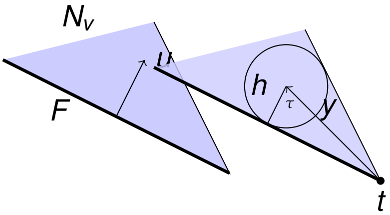

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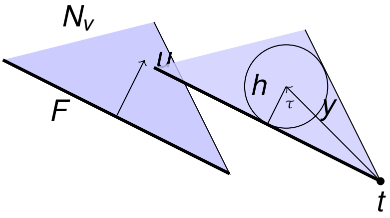

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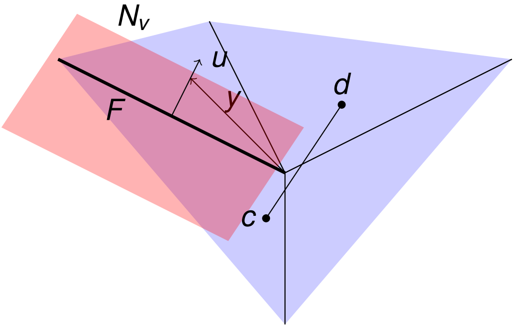
Putting it all together



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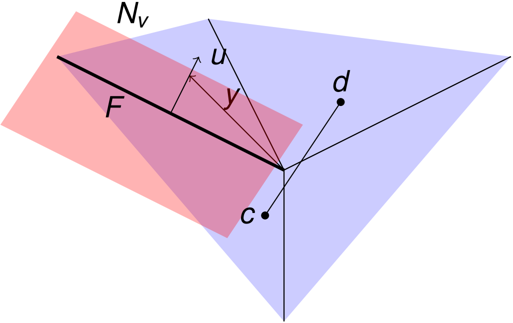

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- Remark: already enough for *weakly polynomial bound*.

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- Solution: can identity optimal facet from w and d'!

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- Remark: Solves open problem of Eisenbrand and Vempala.

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- Recursing n times, optimal solution using $O(n^3/\delta \ln(n/\delta))$ pivots.

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- Feasibility: Use standard reductions to optimization (Phase 1 simplex).

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- Improved diameter bounds and simplex algorithm for *curved polyhedra*.
- Inspired by path finding algorithm over the Voronoi graph of a lattice by Bonifas, D. (2014) used for solving the Closest Vector Problem.

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Thank you!