Exercise 1 (Exact Volume of ℓ_p Balls)
Let $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$, $a > 0$ denote the Gamma function. You will prove that

$$\mathrm{vol}_n(B_p^n) = (2\Gamma(1/p)/p)^n/\Gamma(n/p+1).$$

1. Show that $\int_{\mathbb{R}^n} e^{-\|x\|_p^p}dx = (2\Gamma(1/p)/p)^n$.
(Hint: Split the integral along each coordinate and apply a change of variables.)

2. Show that $\int_{\mathbb{R}^n} e^{-\|x\|_p^p}dx = \mathrm{vol}_n(B_p^n)\Gamma(n/p+1)$.
(Hint: Use the identity $e^{-\|x\|_p^p} = \int_{\|x\|_p}^\infty pt^{p-1}e^{-t^p}dt$)

Exercise 2 (Duality between ℓ_2 norm minimization and ellipsoid maximation)
Let $A \in \mathbb{R}^{m\times n}$ be a non-singular matrix and $b \in \mathbb{R}^n$.

1. Show that

$$\min\{\|x\|_2 : x \in \mathbb{R}^m, A^\mathsf{T}x = b\} \geq \max\{y^\mathsf{T}b : y \in \mathbb{R}^n, \|Ay\|_2 \leq 1\}.$$

(Hint: Use Cauchy-Schwarz.)

2. Show that the value of both systems is equal to $\sqrt{b^\mathsf{T}(A^\mathsf{T}A)^{-1}b}$ and hence both have same value. (Hint: Use the fact that $\|Ay\|_2 \leq 1$ defines an ellipsoid.)

3. Let $A_1, \ldots, A_k$ be matrices where $A_i \in \mathbb{R}^{m_i\times n}$, $i \in [k]$, let $\lambda_1, \ldots, \lambda_k > 0$ and $b \in \mathbb{R}^n$. Assume that $\sum_{i=1}^k \lambda_i A_i^\mathsf{T}A_i \succ 0$ (positive definite). Use the above to show that the systems

$$1.\ \min\{\sqrt{\sum_{i=1}^k \lambda_i\|x_i\|^2} : x_i \in \mathbb{R}^{m_i}, i \in [k], \sum_{i=1}^k \lambda_i A_i^\mathsf{T}x_i = b\}$$

$$2.\ \max\{b^\mathsf{T}y : y \in \mathbb{R}^n, \sum_{i=1}^k \lambda_i\|A_i y\|_2^2 \leq 1\}$$

are strong duals to each other. That is, show that any solution to (1) has value at least that of (2), and the optimal solutions have same value.

Exercise 3 (1/n-Concavity of Determinant) Let $A, B \succ 0$ be $n \times n$ positive definite matrices.

1. Show that $\det(A)^{1/n} = \min\{\mathrm{tr}(AX)/n : X \succ 0, \det(X) = 1\}$, where equality on the right hand side is uniquely attained at $X = A^{-1}\det(A)^{1/n}$. (Hint: Recall that the trace of a matrix is the sum of the eigenvalues while the determinant is the product. Compare the two via the arithmetic mean - geometric mean (AM-GM) inequality.)

2. Conclude that for $\lambda \in (0,1)$, $\det(\lambda A)^{1/n} + (1-\lambda)\det(B)^{1/n} \leq \det(\lambda A + (1-\lambda)B)^{1/n}$ with equality iff $A \in \mathbb{R}_+ B$.

Exercise 4 (Approximation by a Simplex)
Let $K \subset \mathbb{R}^n$ be a convex body. Let $\Delta = \mathrm{conv}(p_1, \ldots, p_{n+1})$ denote a maximum volume simplex contained in K and let $c = \sum_{i=1}^{n+1} p_i/(n+1)$ denote its center.

1. Show that such a simplex exists and give an example body K where it is not unique.

2. For $i \in [n+1]$, let η_i denote a unit normal to the hyperplane $H_i = \text{aff.hull}(p_j : j \in [n+1] \setminus \{i\})$, pointing in the direction of p_i, and let $s_i \in \mathbb{R}^n$ satisfy $H_i = \{x \in \mathbb{R}^n : \langle \eta_i, x \rangle = s_i\}$ (note that by assumption $\langle \eta_i, p_i \rangle > s_i$). Prove that
$$K \subseteq \{x \in \mathbb{R}^n : |\langle \eta_i, x \rangle - s_i| \leq \langle \eta_i, p_i \rangle - s_i\}.$$
(Hint: Show that if not one can replace p_i to make a simplex of larger volume.)

3. Conclude that $\Delta - c \subseteq K - c \subseteq (n+2)(\Delta - c)$.