Geometric Functional Analysis and its Applications – Homework 1

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Exercise 1 (Distance between ℓ_1 and ℓ_∞).
First of all, recall the following result: if $M \in \mathrm{Mat}(\mathbb{R}, m \times n)$ is interpreted as an operator $\ell_1^n \to \ell_\infty^m$, then its operator norm is equal to the maximum absolute value of its entries. There are two ways to prove this:

- For an arbitrary normed space X (not necessarily finite-dimensional) and an operator $T : \ell_1^n \to X$, one has $\|T\|_{\mathrm{op}} = \max\{\|Te_1\|_X, \ldots, \|Te_n\|_X\}$, by a simply application of the triangle inequality. In particular, the operator norm of $M : \ell_1^n \to \ell_\infty^m$ is equal to the largest ℓ^∞-norm of the columns of M, or equivalently, the maximum absolute value of its entries.
- Dually, for an arbitrary normed space X and an operator $S : X \to \ell_\infty^m$, we consider the coordinate projections $\pi_1, \ldots, \pi_m : \ell_\infty^m \to \mathbb{R}$, given by $\pi_i(y_1, \ldots, y_m) = y_i$. Then it is easy to see that $\|S\|_{\mathrm{op}} = \max\{\|\pi_1 \circ S\|_{\mathrm{op}}, \ldots, \|\pi_m \circ S\|_{\mathrm{op}}\}$ holds. In particular, the operator norm of $M : \ell_1^n \to \ell_\infty^m$ is equal to the largest $(\ell^1)^* = \ell^\infty$ norm of the rows of M. Again, this is simply the maximum absolute value of the entries of M.

For $a \in \mathbb{N}_0$, let $\beta(a) = \{\beta(a)_k\}_{k=0}^\infty$ denote the binary expansion of n:

$$a = \sum_{k=0}^{\infty} \beta(a)_k \cdot 2^k.$$

Then $\beta(a)$ is a sequence in $\{0, 1\}$ with at most finitely many non-zero terms. For $a, b \in \mathbb{N}_0$, let $\langle \beta(a), \beta(b) \rangle \in \mathbb{N}_0$ denote the standard inner product of these two sequences:

$$\langle \beta(a), \beta(b) \rangle = \sum_{k=0}^{\infty} \beta(a)_k \cdot \beta(b)_k.$$

(This is well-defined because there are only finitely many non-zero terms.) Recall from lecture 1: for every $k \in \mathbb{N}_0$, the Hadamard matrix $H_{2^k} \in \mathrm{Mat}(\mathbb{R}, 2^k \times 2^k)$ given by

$$(H_{2^k})_{ij} = (-1)^{\langle \beta(i-1), \beta(j-1) \rangle}$$

satisfies $\|H_{2^k}\|_{\ell^1 \to \ell^\infty} = 1$ and $\|H_{2^k}^{-1}\|_{\ell^\infty \to \ell^1} \leq \sqrt{2^k} = 2^{\frac{k}{2}}$.[1] We shall use these matrices to construct for every $n \in \mathbb{N}_1$ a matrix $M_n \in \mathrm{Mat}(\mathbb{R}, n \times n)$ satisfying

$$\|M_n\|_{\ell^1 \to \ell^\infty} \cdot \|M_n^{-1}\|_{\ell^\infty \to \ell^1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{n} = (2 + \sqrt{2}) \cdot \sqrt{n} \leq 3.415 \cdot \sqrt{n}.$$

To that end, let $n \in \mathbb{N}_1$ be fixed. Write $\mathrm{supp}(\beta(n)) := \{k \in \mathbb{N}_0 : \beta(n)_k \neq 0\}$, and define

$$M_n := \bigoplus_{k \in \mathrm{supp}(\beta(n))} H_{2^k}.$$

[1] In the lecture notes, it is only proved that $\|H_{2^k}\|_{\ell^1 \to \ell^\infty} \leq 1$ holds, but the equality follows from the considerations at the beginning of this solution.

Then M_n is a real $n \times n$ matrix with entries in $\{-1, 0, 1\}$. Since $n \geq 1$, we have $\mathrm{supp}(\beta(n)) \neq \varnothing$, so M_n has non-zero entries. Therefore it is clear from the considerations at the beginning of this solution that $\|M_n\|_{\ell^1 \to \ell^\infty} = 1$ holds.

Since M_n is the direct sum of a (non-empty) collection of invertible matrices, we see that M_n itself is also invertible, with inverse

$$M_n^{-1} \;=\; \bigoplus_{k \in \mathrm{supp}(\beta(n))} H_{2^k}^{-1}.$$

Let us write $\mathrm{supp}(\beta(n)) = \{k_1, \ldots, k_s\}$ with $k_1 < k_2 < \cdots < k_s = \lfloor \log_2(n) \rfloor$. Then we can identify $\mathbb{R}^n$ with $\mathbb{R}^{k_1} \oplus \cdots \oplus \mathbb{R}^{k_s}$. In this setting we have

$$M_n^{-1}(x_1 \oplus \cdots \oplus x_s) \;=\; H_{2^{k_1}}^{-1} x_1 \oplus \cdots \oplus H_{2^{k_s}}^{-1} x_s, \qquad (x_i \in \mathbb{R}^{k_i} \text{ for all } i \in [s]).$$

Now let $\vec{x} = x_1 \oplus \cdots \oplus x_s \in \mathbb{R}^n$ be given ($x_i \in \mathbb{R}^{k_i}$ for all $i \in [s]$) satisfying $\|\vec{x}\|_\infty \leq 1$. Then for all $i \in [s]$ we have $\|x_i\|_\infty \leq 1$, hence

$$\left\|H_{2^{k_i}}^{-1} x_i\right\|_1 \;\leq\; \left\|H_{2^{k_i}}^{-1}\right\|_{\ell^\infty \to \ell^1} \cdot 1 \;\leq\; \sqrt{2^{k_i}} \;=\; 2^{\frac{k_i}{2}}.$$

Consequently, we find

$$\begin{aligned}
\left\|M_n^{-1}\vec{x}\right\|_1 \;&=\; \left\|H_{2^{k_1}}^{-1} x_1 \oplus \cdots \oplus H_{2^{k_s}}^{-1} x_s\right\|_1 \\
&=\; \left\|H_{2^{k_1}}^{-1} x_1\right\|_1 + \cdots + \left\|H_{2^{k_s}}^{-1} x_s\right\|_1 \\
&\leq\; 2^{\frac{k_1}{2}} + \cdots + 2^{\frac{k_s}{2}} \\
&\leq\; \sum_{i=0}^{k_s} \sqrt{2}^{\,i} \\
&=\; \frac{\sqrt{2}^{\,k_s+1} - 1}{\sqrt{2} - 1} \\
&\leq\; \frac{\sqrt{2}^{\,k_s+1}}{\sqrt{2} - 1} \\
&=\; \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot 2^{\frac{k_s}{2}} \\
&=\; \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{2^{\lfloor \log_2(n) \rfloor}} \\
&\leq\; \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{2^{\log_2(n)}} \\
&=\; \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{n}.
\end{aligned}$$

As this holds for all $\vec{x} \in \mathbb{R}^n$ satisfying $\|\vec{x}\|_\infty \leq 1$, we find $\|M_n^{-1}\|_{\ell^\infty \to \ell^1} \leq \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \sqrt{n}$. Thus, if we interpret M_n as an operator $\ell_1^n \to \ell_\infty^n$, then we have

$$\|M_n\|_{\mathrm{op}} \cdot \|M_n^{-1}\|_{\mathrm{op}} \;\leq\; 1 \cdot \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{n} \;=\; (2 + \sqrt{2}) \cdot \sqrt{n},$$

as promised. ■

Exercise 2 (Little Grothendieck inequality).

1. Let $A \in \mathrm{Mat}(\mathbb{R}, n \times n)$ be positive semidefinite, and choose some $B \in \mathrm{Mat}(\mathbb{R}, m \times n)$ such that $A = B^\top B$. Define $L_A, R_A \in \mathbb{R}$ by

$$L_A := \max_{x_1,\dots,x_n,y_1,\dots,y_n \in \mathbb{S}^{2n-1}} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle;$$

$$R_A := \max_{x_1,\dots,x_n \in \mathbb{S}^{2n-1}} \sum_{i,j=1}^{n} A_{ij} \langle x_i, x_j \rangle.$$

Then $L_A = \|A\|_G$ is simply the left-hand side of the inequality from the problem statement. On the other hand, R_A has a different domain of maximisation (compared to the right-hand side of the inequality from the problem statement), but this is inconsequential: given the vectors $x_1, \dots, x_n \in \mathbb{S}^{2n-1}$, we can find a Hilbert space isomorphism $\mathrm{span}(x_1, \dots, x_n) \cong \mathbb{R}^d$ for some $d \leq n$, so we may find vectors $x_1', \dots, x_n' \in \mathbb{S}^{n-1}$ with $\langle x_i', x_j' \rangle = \langle x_i, x_j \rangle$ for all $i, j \in [n]$. Therefore we have

$$R_A = \max_{x_1',\dots,x_n' \in \mathbb{S}^{n-1}} \sum_{i,j=1}^{n} A_{ij} \langle x_i', x_j' \rangle,$$

and we find that L_A and R_A are simply the left- and right-hand side maxima from the problem statement.

Clearly we have

$$R_A = \max_{\substack{x_1,\dots,x_n,y_1,\dots,y_n \in \mathbb{S}^{2n-1} \\ x_1=y_1,\dots,x_n=y_n}} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle \leq \max_{x_1,\dots,x_n,y_1,\dots,y_n \in \mathbb{S}^{2n-1}} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle = L_A.$$

The problem at hand is to prove the reverse inequality, $L_A \leq R_A$.

Given vectors $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{S}^{2n-1}$, define the matrices $X, Y \in \mathrm{Mat}(\mathbb{R}, 2n \times n)$ by

$$X := \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}, \qquad \text{and} \qquad Y := \begin{pmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{pmatrix}.$$

Then we have

$$(X^\top Y)_{ij} = \sum_{k=1}^{n} X_{ik}^\top Y_{kj} = \sum_{k=1}^{n} X_{ki} Y_{kj} = \langle x_i, y_j \rangle. \tag{2.1}$$

For arbitrary $m', n' \in \mathbb{N}_1$, let $\langle \cdot\,, \cdot \rangle_{\mathrm{tr}} : \mathrm{Mat}(\mathbb{R}, m' \times n') \to \mathbb{R}_{\geq 0}$ denote the trace inner product

$$\langle C, D \rangle_{\mathrm{tr}} := \mathrm{tr}(D^\top C) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} C_{ij} D_{ij}.$$

(The dimensions m' and n' will be clear from the context.)

By (2.1), we may write

$$
\begin{aligned}
\sum_{i,j=1}^{n} A_{ij}\langle x_i, y_j\rangle &= \langle A, X^\top Y\rangle_{\mathrm{tr}} \\
&= \mathrm{tr}\left((X^\top Y)^\top A\right) \\
&= \mathrm{tr}\left(Y^\top X A\right) \\
&= \mathrm{tr}\left(X A Y^\top\right) \\
&= \mathrm{tr}\left(X B^\top B Y^\top\right) \\
&= \mathrm{tr}\left((B X^\top)^\top B Y^\top\right) \\
&= \left\langle BY^\top, BX^\top \right\rangle_{\mathrm{tr}} \\
&\leq \left|\left\langle BY^\top, BX^\top \right\rangle_{\mathrm{tr}}\right| \\
&\leq \left\langle BY^\top, BY^\top \right\rangle_{\mathrm{tr}}^{\frac{1}{2}} \cdot \left\langle BX^\top, BX^\top \right\rangle_{\mathrm{tr}}^{\frac{1}{2}} \\
&= \left(\sum_{i,j=1}^{n} A_{ij}\langle y_i, y_j\rangle\right)^{\frac{1}{2}} \cdot \left(\sum_{i,j=1}^{n} A_{ij}\langle x_i, x_j\rangle\right)^{\frac{1}{2}} \\
&\leq R_A^{\frac{1}{2}} \cdot R_A^{\frac{1}{2}} \\
&= R_A.
\end{aligned}
$$

This shows that we have $L_A \leq R_A$.

2. First of all, recall that for an arbitrary set of vectors $y_1, \ldots, y_n \in \mathbb{R}^d$, the "Gram" matrix $M \in \mathrm{Mat}(\mathbb{R}, n \times n)$ given by $M_{ij} = \langle y_i, y_j\rangle$ is positive semidefinite. After all, if we let $Y \in \mathrm{Mat}(\mathbb{R}, d \times n)$ denote the matrix

$$
Y := \begin{pmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{pmatrix},
$$

then we may write $M = Y^\top Y$, analogously to (2.1), so M is positive semidefinite.

Now, to solve the exercise, we use the Taylor expansion of arcsin. For all $t \in [-1, 1]$ we have

$$
\arcsin(t) = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} \cdot t^{2k+1},
$$

Consequently, for $t \in [-1, 1]$ we may write

$$
\frac{2}{\pi}\arcsin(t) - \frac{2}{\pi}t = \frac{2}{\pi}\big(\arcsin(t) - t\big) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} \cdot t^{2k+1}. \tag{2.2}
$$

Furthermore, define the sequence $\{M_k\}_{k=1}^{\infty}$ in $\mathrm{Mat}(\mathbb{R}, n \times n)$ by setting

$$
(M_k)_{ij} := \langle x_i, x_j\rangle^k = \langle x_i^{\otimes k}, x_j^{\otimes k}\rangle.
$$

Then for every $k \in \mathbb{N}_1$, we have that M_k is positive semidefinite (since it is a Gram matrix). If B is as in the exercise, then we may write

$$\begin{aligned} B &= \left(\frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle \right)_{i,j=1}^{n} \\ &= \left(\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} \cdot \langle x_i, x_j \rangle^{2k+1} \right)_{i,j=1}^{n} \qquad (2.3) \\ &= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} M_{2k+1}. \qquad (2.4) \end{aligned}$$

A word about convergence: since $x_1, \ldots, x_n$ are assumed to be unit vectors, for all $i, j \in [n]$ we have $|\langle x_i, x_j \rangle| \leq \|x_i\| \|x_j\| \leq 1$. It follows that $\langle x_i, x_j \rangle$ lies in the interval of convergence of (2.2), warranting the expression (2.3). Consequently, the expression (2.4) is warranted because all (Hausdorff) vector space topologies on $\mathbb{R}^{n \times n}$ are equivalent, and each is equivalent to the topology of pointwise convergence.

Since all the scalar coefficients in the series (2.4) are non-negative, it follows that

$$B = \frac{2}{\pi} \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{(2k)!}{4^k (k!)^2 (2k+1)} M_{2k+1} \right)$$

is the limit of an increasing sequence of positive semidefinite matrices. Since the positive semidefinite cone is closed, we conclude that B is positive semidefinite.

3. Let $n \in \mathbb{N}_1$ be given, and let $A \in \mathrm{Mat}(\mathbb{R}, n \times n)$ be positive semidefinite. Consider an arbitrary set of unit vectors $x_1, \ldots, x_n \in \mathbb{S}^{n-1}$, and let $g \sim \mathcal{N}(0, I_n)$ be an n-dimensional standard Gaussian vector. Define 1-dimensional real random variables $a_1, \ldots, a_n$ by setting $a_i := \mathrm{sign}(\langle x_i, g \rangle)$ for all $i \in [n]$. Then we have $-1 \leq a_i \leq 1$ for all $i \in [n]$. Furthermore, by Grothendieck's identity, for all $i, j \in [n]$ we have

$$\mathbb{E}[a_i a_j] = \mathbb{E}\big[\mathrm{sign}(\langle x_i, g \rangle)\, \mathrm{sign}(\langle x_j, g \rangle)\big] = \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle).$$

Hence, by linearity of expectation:

$$\mathbb{E}\left[\sum_{i,j=1}^{n} A_{ij} a_i a_j \right] = \sum_{i,j=1}^{n} A_{ij} \cdot \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle). \qquad (2.5)$$

Define $B \in \mathrm{Mat}(\mathbb{R}, n \times n)$ by

$$B_{ij} := \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle.$$

We showed in part 2 of the exercise that B is positive semidefinite.

Now, since A is also positive semidefinite, we may write

$$\begin{aligned}
\sum_{i,j=1}^{n} A_{ij}B_{ij} &= \langle A, B\rangle_{\mathrm{tr}} \\
&= \mathrm{tr}(B^\top A) \\
&= \mathrm{tr}(BA) && \text{(PSD matrix is symmetric)} \\
&= \mathrm{tr}\left(B^{1/2}B^{1/2}A^{1/2}A^{1/2}\right) && \text{(PSD matrix has unique PSD square root)} \\
&= \mathrm{tr}\left(B^{1/2}A^{1/2}A^{1/2}B^{1/2}\right) && \text{(trace is cyclic)} \\
&= \mathrm{tr}\left((B^{1/2})^\top (A^{1/2})^T A^{1/2}B^{1/2}\right) && \text{(PSD matrix is symmetric)} \\
&= \mathrm{tr}\left((A^{1/2}B^{1/2})^\top A^{1/2}B^{1/2}\right) \\
&= \left\langle A^{1/2}B^{1/2}, A^{1/2}B^{1/2}\right\rangle_{\mathrm{tr}} \\
&\geq 0.
\end{aligned}$$

It follows that

$$\sum_{i,j=1}^{n} A_{ij}\left(\frac{2}{\pi}\arcsin(\langle x_i, x_j\rangle) - \frac{2}{\pi}\langle x_i, x_j\rangle\right) \geq 0,$$

or equivalently

$$\sum_{i,j=1}^{n} A_{ij}\cdot\frac{2}{\pi}\arcsin(\langle x_i, x_j\rangle) \geq \sum_{i,j=1}^{n} A_{ij}\cdot\frac{2}{\pi}\langle x_i, x_j\rangle. \tag{2.6}$$

Putting it all together, we find

$$\begin{aligned}
\frac{2}{\pi}\sum_{i,j=1}^{n} A_{ij}\langle x_i, x_j\rangle &\leq \sum_{i,j=1}^{n} A_{ij}\cdot\frac{2}{\pi}\arcsin(\langle x_i, x_j\rangle) && \text{(by (2.6))} \\
&= \mathbb{E}\left[\sum_{i,j=1}^{n} A_{ij}a_i a_j\right] && \text{(by (2.5))} \\
&\leq \sup\left\{\sum_{i,j=1}^{n} A_{ij}b_i c_j \;:\; b, c \in [-1,1]^n\right\} \\
&= \|A\|_{\ell^\infty\to\ell^1}.
\end{aligned}$$

Hence, by part 1 of the exercise, we have

$$\tfrac{2}{\pi}\|A\|_G = \max_{x_1,\ldots,x_n\in\mathbb{S}^{n-1}} \frac{2}{\pi}\sum_{i,j=1}^{n} A_{ij}\langle x_i, x_j\rangle \leq \|A\|_{\ell^\infty\to\ell^1},$$

or equivalently: $\|A\|_G \leq \frac{\pi}{2}\|A\|_{\ell^\infty\to\ell^1}$. ■

Exercise 3 (Covering with an Asymmetric Convex Body).

1. For fixed $x \in \mathbb{R}^n$, we have

$$\mathrm{vol}_n(B \cap (2x - B)) \;=\; \int_{B\cap(2x-B)} 1\, dy \;=\; \int_B \mathbb{1}[y \in 2x - B]\, dy.$$

Note that we have $y \in 2x - B$ if and only if $2x - y \in B$, if and only if $x - \frac{1}{2}y \in \frac{1}{2}B$, if and only if $x \in \frac{1}{2}y + \frac{1}{2}B$. Furthermore, if $y \in B$, then by convexity of B we have $\frac{1}{2}y + \frac{1}{2}B \subseteq B$. As such, we find

$$\begin{aligned}
\mathop{\mathbb{E}}_{x\in B}\big[\,\mathrm{vol}_n(B \cap (2x - B))\big] &= \frac{1}{\mathrm{vol}_n(B)} \int_B \mathrm{vol}_n(B \cap (2x - B))\, dx \\
&= \frac{1}{\mathrm{vol}_n(B)} \int_B \int_B \mathbb{1}[y \in 2x - B]\, dy\, dx \\
&= \frac{1}{\mathrm{vol}_n(B)} \int_B \int_B \mathbb{1}[x \in \tfrac{1}{2}y + \tfrac{1}{2}B]\, dx\, dy \\
&= \frac{1}{\mathrm{vol}_n(B)} \int_B \mathrm{vol}_n(B \cap (\tfrac{1}{2}y + \tfrac{1}{2}B))\, dy \\
&= \frac{1}{\mathrm{vol}_n(B)} \int_B \mathrm{vol}_n(\tfrac{1}{2}y + \tfrac{1}{2}B)\, dy \\
&= \frac{1}{\mathrm{vol}_n(B)} \int_B \mathrm{vol}_n(\tfrac{1}{2}B)\, dy \\
&= \frac{1}{\mathrm{vol}_n(B)} \cdot \mathrm{vol}_n(\tfrac{1}{2}B) \cdot \mathrm{vol}_n(B) \\
&= \mathrm{vol}_n(\tfrac{1}{2}B) \\
&= \frac{\mathrm{vol}_n(B)}{2^n}.
\end{aligned}$$

Hence, by linearity of expectation:

$$\mathop{\mathbb{E}}_{x\in B}\left[\frac{\mathrm{vol}_n(B \cap (2x - B))}{\mathrm{vol}_n(B)}\right] \;=\; \frac{1}{2^n}. \tag{3.1}$$

2. By (3.1), we may choose some $t \in B$ such that $\mathrm{vol}_n(B \cap (2t - B)) \geq \frac{\mathrm{vol}_n(B)}{2^n}$. Using this t, we define $K \subseteq \mathbb{R}^n$ as

$$K \;:=\; \big(B \cap (2t - B)\big) - t \;=\; (B - t) \cap (t - B). \tag{3.2}$$

Since translations, reflections and intersections of compact convex sets are again compact and convex, it is clear that K is compact and convex. Furthermore, we have

$$\mathrm{vol}_n(K) \;=\; \mathrm{vol}_n(B \cap (2t - B)) \;\geq\; \frac{\mathrm{vol}_n(B)}{2^n} \;>\; 0,$$

so K has non-empty interior. Thirdly, it is clear from (3.2) that K is symmetric. (To spell it out, let $x \in K$ be given, then we may choose $b_1, b_2 \in B$ such that $x = b_1 - t = t - b_2$, and we have $-x = b_2 - t = t - b_1 \in (B - t) \cap (t - B) = K$.) Finally, we have

$$K + t \;=\; B \cap (2t - B) \;\subseteq\; B.$$

In conclusion: K is a symmetric convex body satisfying $\mathrm{vol}_n(K) \geq \frac{\mathrm{vol}_n(B)}{2^n}$ and $K + t \subseteq B$.

3. First of all, if $T \subseteq \mathbb{R}^n$ is such that $|T| = N(A,K)$ and $A \subseteq T+K$, then the set $T' := T - t$ satisfies $|T'| = |T|$ and

$$A \subseteq T + K = T' + t + K \subseteq T' + B,$$

so we have $N(A,B) \leq N(A,K)$. For the upper bound, recall from the lecture that we have

$$N(A,K) \leq P(A, \tfrac{K}{2}) \leq 2^n \frac{\mathrm{vol}_n(A + \frac{K}{2})}{\mathrm{vol}_n(K)}.$$

Then, since we have $K + t \subseteq B$, we find

$$A + \tfrac{K}{2} = A - \tfrac{t}{2} + \tfrac{K+t}{2} \subseteq A - \tfrac{t}{2} + \tfrac{B}{2},$$

hence $\mathrm{vol}_n(A + \frac{K}{2}) \leq \mathrm{vol}_n(A - \frac{t}{2} + \frac{B}{2}) = \mathrm{vol}_n(A + \frac{B}{2})$. On the other hand, the lower bound $\mathrm{vol}_n(K) \geq \frac{\mathrm{vol}_n(B)}{2^n}$ gives us $\frac{1}{\mathrm{vol}_n(K)} \leq \frac{2^n}{\mathrm{vol}_n(B)}$. It follows that

$$N(A,K) \leq 2^n \frac{\mathrm{vol}_n(A + \frac{K}{2})}{\mathrm{vol}_n(K)} \leq 2^n \frac{\mathrm{vol}_n(A + \frac{B}{2})}{\mathrm{vol}_n(K)} \leq 4^n \frac{\mathrm{vol}_n(A + \frac{B}{2})}{\mathrm{vol}_n(B)}.$$

This proves that we have

$$N(A,B) \leq N(A,K) \leq 4^n \frac{\mathrm{vol}_n(A + \frac{B}{2})}{\mathrm{vol}_n(B)}. \tag{3.3}$$

Now, for the special cases, we claim that we have $B + \frac{B}{2} = \frac{3}{2}B$. Indeed:

- On the one hand, if $x \in \frac{3}{2}B$, then we may choose $b \in B$ satisfying $x = \frac{3}{2}b$, so we have $x = b + \frac{1}{2}b \in B + \frac{B}{2}$. This proves the inclusion $\frac{3}{2}B \subseteq B + \frac{B}{2}$.
- On the other hand, if $x \in B + \frac{B}{2}$, then we may choose $b_1, b_2 \in B$ satisfying $x = b_1 + \frac{1}{2}b_2$. Now we have $\frac{2}{3}b_1 + \frac{1}{3}b_2 \in B$, by convexity, hence $x = \frac{3}{2}(\frac{2}{3}b_1 + \frac{1}{3}b_2) \in \frac{3}{2}B$. This proves the reverse inclusion $B + \frac{B}{2} \subseteq \frac{3}{2}B$.

Therefore a straightforward application of (3.3) yields

$$N(B,K) \leq 4^n \frac{\mathrm{vol}_n(B + \frac{B}{2})}{\mathrm{vol}_n(B)} = 4^n \frac{\mathrm{vol}_n(\frac{3}{2}B)}{\mathrm{vol}_n(B)} = 4^n \cdot (\tfrac{3}{2})^n = 6^n.$$

Choose some set $T \subseteq \mathbb{R}^n$ with $|T| \leq 6^n$ and $B \subseteq T + K$. Then we also have

$$-B \subseteq -T - K = -T + K = -T - t + K + t \subseteq -T - t + B,$$

where we use that K is symmetric, and that $K + t \subseteq B$. Consequently, we find

$$\begin{aligned}
\mathrm{vol}_n(B - B + \tfrac{B}{2}) &\leq \mathrm{vol}_n(B - T - t + B + \tfrac{B}{2}) \\
&= \mathrm{vol}_n(\tfrac{5}{2}B - T - t) \\
&\leq |T| \cdot \mathrm{vol}_n(\tfrac{5}{2}B - t) \\
&\leq 6^n \cdot \mathrm{vol}_n(\tfrac{5}{2}B) \\
&= 6^n \cdot (\tfrac{5}{2})^n \cdot \mathrm{vol}_n(B) \\
&= 15^n \cdot \mathrm{vol}_n(B).
\end{aligned}$$

Therefore another application of (3.3) yields

$$N(B - B, B) \leq 4^n \frac{\mathrm{vol}_n(B - B + \frac{B}{2})}{\mathrm{vol}_n(B)} \leq 4^n \cdot 15^n = 60^n. \qquad \blacksquare$$

Exercise 4 (Low Rank Approximation of the Identity).
Let $n \in \mathbb{N}_1$ and $\varepsilon \in (0, \frac{1}{4})$ be fixed. By the Johnson–Lindenstrauss lemma (applied to the vectors $0, e_1, \ldots, e_n \in \mathbb{R}^n$ and the constant $\frac{\varepsilon}{4}$), we may choose a positive integer d and a linear map $T : \mathbb{R}^n \to \mathbb{R}^d$ with the following properties:

(i) $d = O\big(\frac{\log(n+1)}{(\frac{\varepsilon}{4})^2}\big) = O\big(\frac{\log(n)}{\varepsilon^2}\big)$;

(ii) For all $i \in [n]$, one has $1 - \frac{\varepsilon}{4} \le \|Te_i\|_2 \le 1 + \frac{\varepsilon}{4}$;

(iii) For all $i, j \in [n]$ one has

$$(1 - \tfrac{\varepsilon}{4}) \cdot \|e_i - e_j\|_2 \le \|Te_i - Te_j\|_2 \le (1 + \tfrac{\varepsilon}{4}) \cdot \|e_i - e_j\|_2.$$

In particular, for $i, j \in [n]$ with $i \neq j$ we have $\|e_i - e_j\|_2 = \sqrt{2}$, hence

$$\|Te_i - Te_j\|_2 \in \big[\, (1 - \tfrac{\varepsilon}{4}) \cdot \sqrt{2}\,,\, (1 + \tfrac{\varepsilon}{4}) \cdot \sqrt{2}\,\big].$$

We prove the following:

(4.a) For all $i \in [n]$ one has $\langle Te_i, Te_i \rangle \in [1 - \varepsilon, 1 + \varepsilon]$;

(4.b) For all $i, j \in [n]$ with $i \neq j$ one has $\langle Te_i, Te_j \rangle \in [-\varepsilon, \varepsilon]$.

For (4.a), note that we have

$$\langle Te_i, Te_i \rangle = \|Te_i\|_2^2 \le (1 + \tfrac{\varepsilon}{4})^2 = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} \le 1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{16} < 1 + \varepsilon,$$

where we used that $\varepsilon < 1$. Similarly, we have

$$\langle Te_i, Te_i \rangle = \|Te_i\|_2^2 \ge (1 - \tfrac{\varepsilon}{4})^2 = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} \ge 1 - \frac{\varepsilon}{2} > 1 - \varepsilon,$$

so (4.a) is proved. For (4.b), note that for arbitrary $i, j \in [n]$ we may write

$$\langle Te_i - Te_j\,,\, Te_i - Te_j \rangle = \langle Te_i\,,\, Te_i \rangle - 2\langle Te_i\,,\, Te_j \rangle + \langle Te_j\,,\, Te_j \rangle,$$

hence

$$\begin{aligned} \langle Te_i\,,\, Te_j \rangle &= \tfrac{1}{2}\langle Te_i\,,\, Te_i \rangle + \tfrac{1}{2}\langle Te_j\,,\, Te_j \rangle - \tfrac{1}{2}\langle Te_i - Te_j\,,\, Te_i - Te_j \rangle \\ &= \tfrac{1}{2}\|Te_i\|_2^2 + \tfrac{1}{2}\|Te_j\|_2^2 - \tfrac{1}{2}\|Te_i - Te_j\|_2^2. \end{aligned}$$

In particular, if $i \neq j$ then we find

$$\begin{aligned} \langle Te_i\,,\, Te_j \rangle &\le \tfrac{1}{2}(1 + \tfrac{\varepsilon}{4})^2 + \tfrac{1}{2}(1 + \tfrac{\varepsilon}{4})^2 - \tfrac{1}{2}\big((1 - \tfrac{\varepsilon}{4}) \cdot \sqrt{2}\big)^2 \\ &= (1 + \tfrac{\varepsilon}{4})^2 - (1 - \tfrac{\varepsilon}{4})^2 \\ &= \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16}\right) - \left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16}\right) \\ &= \varepsilon, \end{aligned}$$

and similarly

$$\begin{aligned}
\langle Te_i\,,\,Te_j\rangle \;&\geq\; \tfrac{1}{2}(1-\tfrac{\varepsilon}{4})^2 + \tfrac{1}{2}(1-\tfrac{\varepsilon}{4})^2 - \tfrac{1}{2}\big((1+\tfrac{\varepsilon}{4})\cdot\sqrt{2}\big)^2 \\
&=\; (1-\tfrac{\varepsilon}{4})^2 - (1+\tfrac{\varepsilon}{4})^2 \\
&=\; \left(1-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{16}\right) - \left(1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{16}\right) \\
&=\; -\varepsilon.
\end{aligned}$$

This proves (4.b).

Now let $B \in \mathrm{Mat}(\mathbb{R}, d \times n)$ be the matrix

$$B \;:=\; \begin{pmatrix} | & & | \\ Te_1 & \cdots & Te_n \\ | & & | \end{pmatrix},$$

and define $\tilde{I}_n := B^\top B$. As we saw in (2.1), for all $i, j \in [n]$ we have $(\tilde{I}_n)_{ij} = \langle Te_i, Te_j\rangle$, so it follows from (4.a) and (4.b) that $|(I_n - \tilde{I}_n)_{ij}| \leq \varepsilon$ holds for all $i, j \in [n]$. Furthermore, $\tilde{I}_n$ is positive semidefinite as it is of the form $B^\top B$. Finally, note that we have

$$\mathrm{rank}(\tilde{I}_n) \;\leq\; \min\big(\mathrm{rank}(B^\top), \mathrm{rank}(B)\big) \;=\; \mathrm{rank}(B) \;\leq\; \min(d, n) \;\leq\; d,$$

and it was established before that d is $O(\frac{\log(n)}{\varepsilon^2})$. ■