Exercise 1 (Distance between $\ell_1$ and $\ell_\infty$).

First of all, recall the following result: if $M \in \text{Mat}(\mathbb{R}, m \times n)$ is interpreted as an operator $\ell_1^m \to \ell_\infty^n$, then its operator norm is equal to the maximum absolute value of its entries. There are two ways to prove this:

- For an arbitrary normed space $X$ (not necessarily finite-dimensional) and an operator $T : \ell_1^m \to X$, one has $\|T\|_{\text{op}} = \max\{\|Te_1\|_X, \ldots, \|Te_n\|_X\}$, by a simply application of the triangle inequality. In particular, the operator norm of $M : \ell_1^m \to \ell_\infty^n$ is equal to the largest $\ell_\infty$-norm of the columns of $M$, or equivalently, the maximum absolute value of its entries.

- Dually, for an arbitrary normed space $X$ and an operator $S : X \to \ell_\infty^n$, we consider the coordinate projections $\pi_1, \ldots, \pi_m : \ell_\infty^n \to \mathbb{R}$, given by $\pi_i(y_1, \ldots, y_m) = y_i$. Then it is easy to see that $\|S\|_{\text{op}} = \max\{\|\pi_1 \circ S\|, \ldots, \|\pi_m \circ S\|\}$ holds. In particular, the operator norm of $M : \ell_1^m \to \ell_\infty^n$ is equal to the largest $(\ell_1)^* = \ell_\infty$ norm of the rows of $M$. Again, this is simply the maximum absolute value of the entries of $M$.

For $a \in \mathbb{N}_0$, let $\beta(a) = \{\beta(a)_k\}_{k=0}^{\infty}$ denote the binary expansion of $n$:

$$a = \sum_{k=0}^{\infty} \beta(a)_k \cdot 2^k.$$  

Then $\beta(a)$ is a sequence in $\{0, 1\}$ with at most finitely many non-zero terms. For $a, b \in \mathbb{N}_0$, let $\langle \beta(a), \beta(b) \rangle \in \mathbb{N}_0$ denote the standard inner product of these two sequences:

$$\langle \beta(a), \beta(b) \rangle = \sum_{k=0}^{\infty} \beta(a)_k \cdot \beta(b)_k.$$  

(This is well-defined because there are only finitely many non-zero terms.) Recall from lecture 1: for every $k \in \mathbb{N}_0$, the Hadamard matrix $H_{2^k} \in \text{Mat}(\mathbb{R}, 2^k \times 2^k)$ given by

$$(H_{2^k})_{ij} = (-1)^{\beta(i-1) \cdot \beta(j-1)}$$

satisfies $\|H_{2^k}\|_{\ell_1 \to \ell_\infty} = 1$ and $\|H_{2^k}^{-1}\|_{\ell_\infty \to \ell_1} \leq \sqrt{2^k} = 2^{\frac{k}{2}}$ \footnote{In the lecture notes, it is only proved that $\|H_{2^k}\|_{\ell_1 \to \ell_\infty} \leq 1$ holds, but the equality follows from the considerations at the beginning of this solution.}. We shall use these matrices to construct for every $n \in \mathbb{N}_1$ a matrix $M_n \in \text{Mat}(\mathbb{R}, n \times n)$ satisfying

$$\|M_n\|_{\ell_1 \to \ell_\infty} \cdot \|M_n^{-1}\|_{\ell_\infty \to \ell_1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \sqrt{n} = (2 + \sqrt{2}) \cdot \sqrt{n} \leq 3.415 \cdot \sqrt{n}.$$  

To that end, let $n \in \mathbb{N}_1$ be fixed. Write $\text{supp}(\beta(n)) := \{k \in \mathbb{N}_0 : \beta(n)_k \neq 0\}$, and define

$$M_n := \bigoplus_{k \in \text{supp}(\beta(n))} H_{2^k}.$$
Then $M_n$ is a real $n \times n$ matrix with entries in $\{-1, 0, 1\}$. Since $n \geq 1$, we have $\text{supp}(\beta(n)) \neq \emptyset$, so $M_n$ has non-zero entries. Therefore it is clear from the considerations at the beginning of this solution that $\|M_n\|_{\ell^1 \to \ell^\infty} = 1$ holds.

Since $M_n$ is the direct sum of a (non-empty) collection of invertible matrices, we see that $M_n$ itself is also invertible, with inverse

$$M_n^{-1} = \bigoplus_{k \in \text{supp}(\beta(n))} H_{2^k}^{-1}.$$  

Let us write $\text{supp}(\beta(n)) = \{k_1, \ldots, k_s\}$ with $k_1 < k_2 < \cdots < k_s = \lceil \log_2(n) \rceil$. Then we can identify $\mathbb{R}^n$ with $\mathbb{R}^{k_1} \oplus \cdots \oplus \mathbb{R}^{k_s}$. In this setting we have

$$M_n^{-1}(x_1 \oplus \cdots \oplus x_s) = H_{2^{k_1}}^{-1}x_1 \oplus \cdots \oplus H_{2^{k_s}}^{-1}x_s, \quad (x_i \in \mathbb{R}^{k_i} \text{ for all } i \in [s]).$$

Now let $\vec{x} = x_1 \oplus \cdots \oplus x_s \in \mathbb{R}^n$ be given $(x_i \in \mathbb{R}^{k_i} \text{ for all } i \in [s])$ satisfying $\|\vec{x}\|_\infty \leq 1$. Then for all $i \in [s]$ we have $\|x_i\|_\infty \leq 1$, hence

$$\begin{align*}
\|H_{2^{k_i}}^{-1}x_i\|_1 &\leq \|H_{2^{k_i}}^{-1}\|_{\ell^\infty \to \ell^1} \cdot 1 \leq \sqrt{2^{k_i}} = 2^{\frac{k_i}{2}}.
\end{align*}$$

Consequently, we find

$$\begin{align*}
\|M_n^{-1}\vec{x}\|_1 &= \|H_{2^{k_1}}^{-1}x_1 \oplus \cdots \oplus H_{2^{k_s}}^{-1}x_s\|_1 \\
&= \|H_{2^{k_1}}^{-1}x_1\|_1 + \cdots + \|H_{2^{k_s}}^{-1}x_s\|_1 \\
&\leq 2^{\frac{k_1}{2}} + \cdots + 2^{\frac{k_s}{2}} \\
&\leq \sum_{i=0}^{k_s} \sqrt{2^i} \\
&= \frac{2^{k_s+1} - 1}{2 - 1} \\
&\leq \frac{\sqrt{2}}{2 - 1} \cdot 2^{\frac{k_s}{2}} \\
&= \frac{\sqrt{2}}{2 - 1} \cdot \sqrt{2^{\lceil \log_2(n) \rceil}} \\
&\leq \frac{\sqrt{2}}{2 - 1} \cdot \sqrt{2^{\log_2(n)}} \\
&= \frac{\sqrt{2}}{2 - 1} \cdot \sqrt{n}.
\end{align*}$$

As this holds for all $\vec{x} \in \mathbb{R}^n$ satisfying $\|\vec{x}\|_\infty \leq 1$, we find $\|M_n^{-1}\|_{\ell^\infty \to \ell^1} \leq \frac{\sqrt{2}}{2 - 1} \cdot \sqrt{n}$. Thus, if we interpret $M_n$ as an operator $\ell^1_n \to \ell'^{\infty}_n$, then we have

$$\|M_n\|_{\text{op}} \cdot \|M_n^{-1}\|_{\text{op}} \leq 1 \cdot \frac{\sqrt{2}}{2 - 1} \cdot \sqrt{n} = (2 + \sqrt{2}) \cdot \sqrt{n},$$

as promised.  

\[\square\]
Exercise 2 (Little Grothendieck inequality).

1. Let $A \in \text{Mat}(\mathbb{R}, n \times n)$ be positive semidefinite, and choose some $B \in \text{Mat}(\mathbb{R}, m \times n)$ such that $A = B^T B$. Define $L_A, R_A \in \mathbb{R}$ by

\[
L_A := \max_{x_1, \ldots, x_n, y_1, \ldots, y_n} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle;
\]

\[
R_A := \max_{x_1, \ldots, x_n} \sum_{i,j=1}^{n} A_{ij} \langle x_i, x_j \rangle.
\]

Then $L_A = \|A\|_G$ is simply the left-hand side of the inequality from the problem statement. On the other hand, $R_A$ has a different domain of maximisation (compared to the right-hand side of the inequality from the problem statement), but this is inconsequential: given the vectors $x_1, \ldots, x_n \in S^{2n-1}$, we can find a Hilbert space isomorphism $\text{span}(x_1, \ldots, x_n) \cong \mathbb{R}^d$ for some $d \leq n$, so we may find vectors $x_1', \ldots, x_n' \in S^{n-1}$ with $\langle x_i', x_j' \rangle = \langle x_i, x_j \rangle$ for all $i, j \in [n]$. Therefore we have

\[
R_A = \max_{x_1', \ldots, x_n'} \sum_{i,j=1}^{n} A_{ij} \langle x_i', x_j' \rangle,
\]

and we find that $L_A$ and $R_A$ are simply the left- and right-hand side maxima from the problem statement.

Clearly we have

\[
R_A = \max_{x_1, \ldots, x_n, y_1, \ldots, y_n} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle \leq \max_{x_1, \ldots, x_n, y_1, \ldots, y_n} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle = L_A.
\]

The problem at hand is to prove the reverse inequality, $L_A \leq R_A$.

Given vectors $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^{2n-1}$, define the matrices $X, Y \in \text{Mat}(\mathbb{R}, 2n \times n)$ by

\[
X := \begin{pmatrix} | & | \\ x_1 & \cdots & x_n \end{pmatrix}, \quad \text{and} \quad Y := \begin{pmatrix} | & | \\ y_1 & \cdots & y_n \end{pmatrix}.
\]

Then we have

\[
(X^T Y)_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj} = \sum_{k=1}^{n} X_{ki} Y_{kj} = \langle x_i, y_j \rangle.
\]

(2.1)

For arbitrary $m', n' \in \mathbb{N}_1$, let $\langle \cdot, \cdot \rangle_{\text{tr}} : \text{Mat}(\mathbb{R}, m' \times n') \to \mathbb{R}_{\geq 0}$ denote the trace inner product

\[
\langle C, D \rangle_{\text{tr}} := \text{tr}(D^T C) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} C_{ij} D_{ij}.
\]

(The dimensions $m'$ and $n'$ will be clear from the context.)
By (2.1), we may write

\[
\sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle = \langle A, X^T Y \rangle_{tr}
\]

\[
= \text{tr} \left( (X^T Y)^T A \right)
\]

\[
= \text{tr} \left( Y^T X A \right)
\]

\[
= \text{tr} \left( X A Y^T \right)
\]

\[
= \text{tr} \left( (B X^T)^T BY^T \right)
\]

\[
= \langle BY^T, B X^T \rangle_{tr}
\]

\[
\leq \left| \langle BY^T, B X^T \rangle_{tr} \right|
\]

\[
\leq \langle BY^T, BY^T \rangle_{tr}^{\frac{1}{2}} \cdot \langle B X^T, B X^T \rangle_{tr}^{\frac{1}{2}}
\]

\[
= \left( \sum_{i,j=1}^{n} A_{ij} \langle x_i, x_j \rangle \right)^{\frac{1}{2}} \cdot \left( \sum_{i,j=1}^{n} A_{ij} \langle y_i, y_j \rangle \right)^{\frac{1}{2}}
\]

\[
\leq R_A^2 \cdot R_A^2
\]

\[
= R_A.
\]

This shows that we have \( L_A \leq R_A \).

2. First of all, recall that for an arbitrary set of vectors \( y_1, \ldots, y_n \in \mathbb{R}^d \), the “Gram” matrix \( M \in \text{Mat}(\mathbb{R}, n \times n) \) given by \( M_{ij} = \langle y_i, y_j \rangle \) is positive semidefinite. After all, if we let \( Y \in \text{Mat}(\mathbb{R}, d \times n) \) denote the matrix

\[
Y := \begin{pmatrix}
| & & \\
y_1 & \cdots & y_n \\
| & & 
\end{pmatrix},
\]

then we may write \( M = Y^T Y \), analogously to (2.1), so \( M \) is positive semidefinite.

Now, to solve the exercise, we use the Taylor expansion of \( \arcsin \). For all \( t \in [-1, 1] \) we have

\[
\arcsin(t) = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k + 1)} t^{2k+1},
\]

Consequently, for \( t \in [-1, 1] \) we may write

\[
\frac{2}{\pi} \arcsin(t) - \frac{2}{\pi} t = \frac{2}{\pi} \left( \arcsin(t) - t \right) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k + 1)} t^{2k+1}.
\] (2.2)

Furthermore, define the sequence \( \{ M_k \}_{k=1}^{\infty} \) in \( \text{Mat}(\mathbb{R}, n \times n) \) by setting

\[
(M_k)_{ij} := (x_i, x_j)^k = (x_i \otimes k, x_j \otimes k).
\]
Then for every \( k \in \mathbb{N} \), we have that \( M_k \) is positive semidefinite (since it is a Gram matrix). If \( B \) is as in the exercise, then we may write

\[
B = \left( \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle \right)_{i,j=1}^n
\]

\[
= \left( \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{4^k(k!)^2(2k+1)} \cdot (x_i, x_j)^{2k+1} \right)_{i,j=1}^n
\]

\[
= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{4^k(k!)^2(2k+1)} M_{2k+1}.
\]  

(2.3)

(2.4)

A word about convergence: since \( x_1, \ldots, x_n \) are assumed to be unit vectors, for all \( i, j \in [n] \) we have \( |\langle x_i, x_j \rangle| \leq \|x_i\| \|x_j\| \leq 1 \). It follows that \( \langle x_i, x_j \rangle \) lies in the interval of convergence of (2.2), warranting the expression (2.3). Consequently, the expression (2.4) is warranted because all (Hausdorff) vector space topologies on \( \mathbb{R}^{n \times n} \) are equivalent, and each is equivalent to the topology of pointwise convergence.

Since all the scalar coefficients in the series (2.4) are non-negative, it follows that

\[
B = \frac{2}{\pi} \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{(2k)!}{4^k(k!)^2(2k+1)} M_{2k+1} \right)
\]

is the limit of an increasing sequence of positive semidefinite matrices. Since the positive semidefinite cone is closed, we conclude that \( B \) is positive semidefinite.

3. Let \( n \in \mathbb{N} \) be given, and let \( A \in \text{Mat}(\mathbb{R}, n \times n) \) be positive semidefinite. Consider an arbitrary set of unit vectors \( x_1, \ldots, x_n \in \mathbb{S}^{n-1} \), and let \( g \sim \mathcal{N}(0, I_n) \) be an \( n \)-dimensional standard Gaussian vector. Define 1-dimensional real random variables \( a_1, \ldots, a_n \) by setting \( a_i := \text{sign}(\langle x_i, g \rangle) \) for all \( i \in [n] \). Then we have \(-1 \leq a_i \leq 1\) for all \( i \in [n] \). Furthermore, by Grothendieck’s identity, for all \( i, j \in [n] \) we have

\[
\mathbb{E}[a_i a_j] = \mathbb{E}\left[ \text{sign}(\langle x_i, g \rangle) \text{sign}(\langle x_j, g \rangle) \right] = \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle).
\]

Hence, by linearity of expectation:

\[
\mathbb{E}\left[ \sum_{i,j=1}^{n} A_{ij} a_i a_j \right] = \sum_{i,j=1}^{n} A_{ij} \cdot \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle).
\]  

(2.5)

Define \( B \in \text{Mat}(\mathbb{R}, n \times n) \) by

\[
B_{ij} := \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle.
\]

We showed in part 2 of the exercise that \( B \) is positive semidefinite.
Now, since $A$ is also positive semidefinite, we may write
\[
\sum_{i,j=1}^{n} A_{ij}B_{ij} = \langle A, B \rangle_{\text{tr}} \\
= \text{tr}(B^T A) \\
= \text{tr}(BA) \quad \text{(PSD matrix is symmetric)}
\]
\[
= \text{tr}(B^{1/2}B^{1/2}A^{1/2}A^{1/2}) \quad \text{(PSD matrix has unique PSD square root)}
\]
\[
= \text{tr}(B^{1/2}A^{1/2}A^{1/2}B^{1/2}) \quad \text{(trace is cyclic)}
\]
\[
= \text{tr}((B^{1/2})^T(A^{1/2})^{T}A^{1/2}B^{1/2}) \quad \text{(PSD matrix is symmetric)}
\]
\[
= \text{tr}((A^{1/2}B^{1/2})^T A^{1/2}B^{1/2})
\]
\[
= \langle A^{1/2}B^{1/2}, A^{1/2}B^{1/2} \rangle_{\text{tr}}
\]
\[
\geq 0.
\]
It follows that
\[
\sum_{i,j=1}^{n} A_{ij} \left( \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle \right) \geq 0,
\]
or equivalently
\[
\sum_{i,j=1}^{n} A_{ij} \cdot \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) \geq \sum_{i,j=1}^{n} A_{ij} \cdot \frac{2}{\pi} \langle x_i, x_j \rangle. \quad (2.6)
\]
Putting it all together, we find
\[
\frac{2}{\pi} \sum_{i,j=1}^{n} A_{ij} \langle x_i, x_j \rangle \leq \sum_{i,j=1}^{n} A_{ij} \cdot \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) \quad \text{(by } (2.6))
\]
\[
= \mathbb{E} \left[ \sum_{i,j=1}^{n} A_{ij}a_ia_j \right] \quad \text{(by } (2.5))
\]
\[
\leq \sup \left\{ \sum_{i,j=1}^{n} A_{ij}b_ic_j : b,c \in [-1,1]^n \right\}
\]
\[
= \|A\|_{\ell^\infty \to \ell^1}.
\]
Hence, by part 1 of the exercise, we have
\[
\frac{2}{\pi} \|A\|_{G} = \max_{x_1,\ldots,x_n \in \mathbb{S}^{n-1}} \frac{2}{\pi} \sum_{i,j=1}^{n} A_{ij} \langle x_i, x_j \rangle \leq \|A\|_{\ell^\infty \to \ell^1},
\]
or equivalently: $\|A\|_{G} \leq \frac{\pi}{2} \|A\|_{\ell^\infty \to \ell^1}$. \hfill \blacksquare
Exercise 3 (Covering with an Asymmetric Convex Body).

1. For fixed \( x \in \mathbb{R}^n \), we have

\[
\text{vol}_n(B \cap (2x - B)) = \int_{B \cap (2x - B)} 1 \, dy = \int_B \mathbb{1}[y \in 2x - B] \, dy.
\]

Note that we have \( y \in 2x - B \) if and only if \( 2x - y \in B \), if and only if \( x - \frac{1}{2}y \in \frac{1}{2}B \), if and only if \( x \in \frac{1}{2}y + \frac{1}{2}B \). Furthermore, if \( y \in B \), then by convexity of \( B \) we have \( \frac{1}{2}y + \frac{1}{2}B \subseteq B \).

As such, we find

\[
\mathbb{E}_{x \in B} \left[ \text{vol}_n(B \cap (2x - B)) \right] = \frac{1}{\text{vol}_n(B)} \int_B \text{vol}_n(B \cap (2x - B)) \, dx
\]

\[
= \frac{1}{\text{vol}_n(B)} \int_B \int_B \mathbb{1}[y \in 2x - B] \, dy \, dx
\]

\[
= \frac{1}{\text{vol}_n(B)} \int_B \int_B \mathbb{1}[x \in \frac{1}{2}y + \frac{1}{2}B] \, dx \, dy
\]

\[
= \frac{1}{\text{vol}_n(B)} \int_B \text{vol}_n(B \cap (\frac{1}{2}y + \frac{1}{2}B)) \, dy
\]

\[
= \frac{1}{\text{vol}_n(B)} \int_B \text{vol}_n(\frac{1}{2}y + \frac{1}{2}B) \, dy
\]

\[
= \frac{1}{\text{vol}_n(B)} \cdot \text{vol}_n(\frac{1}{2}B) \cdot \text{vol}_n(B)
\]

\[
= \text{vol}_n(\frac{1}{2}B)
\]

\[
= \frac{\text{vol}_n(B)}{2^n}.
\]

Hence, by linearity of expectation:

\[
\mathbb{E}_{x \in B} \left[ \frac{\text{vol}_n(B \cap (2x - B))}{\text{vol}_n(B)} \right] = \frac{1}{2^n}. \quad (3.1)
\]

2. By (3.1), we may choose some \( t \in B \) such that \( \text{vol}_n(B \cap (2t - B)) \geq \frac{\text{vol}_n(B)}{2^n} \). Using this \( t \), we define \( K \subseteq \mathbb{R}^n \) as

\[
K := (B \cap (2t - B)) - t = (B - t) \cap (t - B). \quad (3.2)
\]

Since translations, reflections and intersections of compact convex sets are again compact and convex, it is clear that \( K \) is compact and convex. Furthermore, we have

\[
\text{vol}_n(K) = \text{vol}_n(B \cap (2t - B)) \geq \frac{\text{vol}_n(B)}{2^n} > 0,
\]

so \( K \) has non-empty interior. Thirdly, it is clear from (3.2) that \( K \) is symmetric. (To spell it out, let \( x \in K \) be given, then we may choose \( b_1, b_2 \in B \) such that \( x = b_1 - t = t - b_2 \), and we have \( -x = b_2 - t = t - b_1 \in (B - t) \cap (t - B) = K \).) Finally, we have

\[
K + t = B \cap (2t - B) \subseteq B.
\]

In conclusion: \( K \) is a symmetric convex body satisfying \( \text{vol}_n(K) \geq \frac{\text{vol}_n(B)}{2^n} \) and \( K + t \subseteq B \).
3. First of all, if \( T \subseteq \mathbb{R}^n \) is such that \( |T| = N(A, K) \) and \( A \subseteq T + K \), then the set \( T' := T - t \) satisfies \( |T'| = |T| \) and

\[
A \subseteq T + K = T' + t + K \subseteq T' + B,
\]

so we have \( N(A, B) \leq N(A, K) \). For the upper bound, recall from the lecture that we have

\[
N(A, K) \leq P(A, \frac{K}{2}) \leq 2^n \frac{\text{vol}_n(A + \frac{K}{2})}{\text{vol}_n(K)}.
\]

Then, since we have \( K + t \subseteq B \), we find

\[
A + \frac{K}{2} = A - \frac{t}{2} + \frac{K + t}{2} \subseteq A - \frac{t}{2} + \frac{B}{2},
\]

hence \( \text{vol}_n(A + \frac{K}{2}) \leq \text{vol}_n(A - \frac{t}{2} + \frac{B}{2}) = \text{vol}_n(A + \frac{B}{2}) \). On the other hand, the lower bound \( \text{vol}_n(K) \geq \frac{\text{vol}_n(B)}{2^n} \) gives us \( \frac{\text{vol}_n(A + \frac{K}{2})}{\text{vol}_n(K)} \leq \frac{\text{vol}_n(A + \frac{B}{2})}{\text{vol}_n(B)} \). It follows that

\[
N(A, K) \leq 2^n \frac{\text{vol}_n(A + \frac{K}{2})}{\text{vol}_n(K)} \leq 2^n \frac{\text{vol}_n(A + \frac{B}{2})}{\text{vol}_n(K)} \leq 4^n \frac{\text{vol}_n(A + \frac{B}{2})}{\text{vol}_n(B)}.
\]

This proves that we have

\[
N(A, B) \leq N(A, K) \leq 4^n \frac{\text{vol}_n(A + \frac{B}{2})}{\text{vol}_n(B)}. \tag{3.3}
\]

Now, for the special cases, we claim that we have \( B + \frac{B}{2} = \frac{3}{2}B \). Indeed:

- On the one hand, if \( x \in \frac{3}{2}B \), then we may choose \( b \in B \) satisfying \( x = \frac{3}{2}b \), so we have \( x = b + \frac{1}{2}b \in B + \frac{B}{2} \). This proves the inclusion \( \frac{3}{2}B \subseteq B + \frac{B}{2} \).

- On the other hand, if \( x \in B + \frac{B}{2} \), then we may choose \( b_1, b_2 \in B \) satisfying \( x = b_1 + \frac{1}{2}b_2 \). Now we have \( \frac{3}{2}b_1 + \frac{1}{2}b_2 \in \frac{3}{2}B \), by convexity, hence \( x = \frac{3}{2}(\frac{3}{2}b_1 + \frac{1}{2}b_2) \in \frac{3}{2}B \). This proves the reverse inclusion \( B + \frac{B}{2} \subseteq \frac{3}{2}B \).

Therefore a straightforward application of (3.3) yields

\[
N(B, K) \leq 4^n \frac{\text{vol}_n(B + \frac{B}{2})}{\text{vol}_n(B)} = 4^n \frac{\text{vol}_n(\frac{3}{2}B)}{\text{vol}_n(B)} = 4^n \cdot (\frac{3}{2})^n = 6^n.
\]

Choose some set \( T \subseteq \mathbb{R}^n \) with \( |T| \leq 6^n \) and \( B \subseteq T + K \). Then we also have

\[
-B \subseteq -T - K = -T + K = -T - t + K + t \subseteq -T - t + B,
\]

where we use that \( K \) is symmetric, and that \( K + t \subseteq B \). Consequently, we find

\[
\text{vol}_n(B - B + \frac{B}{2}) \leq \text{vol}_n(B - T - t + B + \frac{B}{2}) = \text{vol}_n(\frac{5}{2}B - T - t) \leq |T| \cdot \text{vol}_n(\frac{5}{2}B - t) \leq 6^n \cdot \text{vol}_n(\frac{5}{2}B) = 6^n \cdot (\frac{5}{2})^n \cdot \text{vol}_n(B) = 15^n \cdot \text{vol}_n(B).
\]

Therefore another application of (3.3) yields

\[
N(B - B, B) \leq 4^n \frac{\text{vol}_n(B - B + \frac{B}{2})}{\text{vol}_n(B)} \leq 4^n \cdot 15^n = 60^n.
\]

\[\blacksquare\]
Exercise 4 (Low Rank Approximation of the Identity).

Let \( n \in \mathbb{N}_1 \) and \( \varepsilon \in (0, \frac{1}{4}) \) be fixed. By the Johnson–Lindenstrauss lemma (applied to the vectors \( 0, e_1, \ldots, e_n \in \mathbb{R}^n \) and the constant \( \frac{\varepsilon}{4} \)), we may choose a positive integer \( d \) and a linear map \( T : \mathbb{R}^n \to \mathbb{R}^d \) with the following properties:

(i) \( d = O\left(\frac{\log(n+1)}{(\varepsilon/4)^2}\right) = O\left(\frac{\log(n)}{\varepsilon^2}\right) \);

(ii) For all \( i \in [n] \), one has \( 1 - \frac{\varepsilon}{4} \leq \|Te_i\|_2 \leq 1 + \frac{\varepsilon}{4} \);

(iii) For all \( i, j \in [n] \) one has

\[
(1 - \frac{\varepsilon}{4}) \cdot \|e_i - e_j\|_2 \leq \|Te_i - Te_j\|_2 \leq (1 + \frac{\varepsilon}{4}) \cdot \|e_i - e_j\|_2.
\]

In particular, for \( i, j \in [n] \) with \( i \neq j \) we have \( \|e_i - e_j\|_2 = \sqrt{2} \), hence

\[
\|Te_i - Te_j\|_2 \in \left[ (1 - \frac{\varepsilon}{4}) \cdot \sqrt{2}, (1 + \frac{\varepsilon}{4}) \cdot \sqrt{2} \right].
\]

We prove the following:

(4.a) For all \( i \in [n] \) one has \( \langle Te_i, Te_i \rangle \in [1 - \varepsilon, 1 + \varepsilon] \);

(4.b) For all \( i, j \in [n] \) with \( i \neq j \) one has \( \langle Te_i, Te_j \rangle \in [-\varepsilon, \varepsilon] \).

For (4.a) note that we have

\[
\langle Te_i, Te_i \rangle = \|Te_i\|_2^2 \leq (1 + \frac{\varepsilon}{4})^2 = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} \leq 1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{16} < 1 + \varepsilon,
\]

where we used that \( \varepsilon < 1 \). Similarly, we have

\[
\langle Te_i, Te_i \rangle = \|Te_i\|_2^2 \geq (1 - \frac{\varepsilon}{4})^2 = 1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{16} \geq 1 - \frac{\varepsilon}{2} > 1 - \varepsilon,
\]

so (4.a) is proved. For (4.b) note that for arbitrary \( i, j \in [n] \) we may write

\[
\langle Te_i - Te_j, Te_i - Te_j \rangle = \langle Te_i, Te_i \rangle - 2\langle Te_i, Te_j \rangle + \langle Te_j, Te_j \rangle,
\]

hence

\[
\langle Te_i, Te_j \rangle = \frac{1}{2}\langle Te_i, Te_i \rangle + \frac{1}{2}\langle Te_j, Te_j \rangle - \frac{1}{2}\langle Te_i - Te_j, Te_i - Te_j \rangle
\]

\[
= \frac{1}{2}\|Te_i\|_2^2 + \frac{1}{2}\|Te_j\|_2^2 - \frac{1}{2}\|Te_i - Te_j\|_2^2.
\]

In particular, if \( i \neq j \) then we find

\[
\langle Te_i, Te_j \rangle \leq \frac{1}{2}(1 + \frac{\varepsilon}{4})^2 + \frac{1}{2}(1 + \frac{\varepsilon}{4})^2 - \frac{1}{2}(1 - \frac{\varepsilon}{4})^2 \cdot \sqrt{2}
\]

\[
= (1 + \frac{\varepsilon}{2})^2 - (1 - \frac{\varepsilon}{4})^2
\]

\[
= \left( 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} \right) - \left( 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} \right)
\]

\[
= \varepsilon,
\]
and similarly
\[
\langle Te_i, Te_j \rangle \geq \frac{1}{2}(1 - \frac{\varepsilon}{4})^2 + \frac{1}{2}(1 - \frac{\varepsilon}{4})^2 - \frac{1}{2}((1 + \frac{\varepsilon}{4}) \cdot \sqrt{2})^2
\]
\[
= (1 - \frac{\varepsilon}{4})^2 - (1 + \frac{\varepsilon}{4})^2
\]
\[
= \left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16}\right) - \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16}\right)
\]
\[
= -\varepsilon.
\]
This proves (4.b).

Now let \( B \in \text{Mat}(\mathbb{R}, d \times n) \) be the matrix
\[
B := \begin{pmatrix}
Te_1 & \cdots & Te_n
\end{pmatrix},
\]
and define \( \tilde{I}_n := B^T B \). As we saw in (2.1), for all \( i, j \in [n] \) we have \( (\tilde{I}_n)_{ij} = \langle Te_i, Te_j \rangle \), so it follows from (4.a) and (4.b) that \( |(I_n - \tilde{I}_n)_{ij}| \leq \varepsilon \) holds for all \( i, j \in [n] \). Furthermore, \( \tilde{I}_n \) is positive semidefinite as it is of the form \( B^T B \). Finally, note that we have
\[
\text{rank}(\tilde{I}_n) \leq \min(\text{rank}(B^T), \text{rank}(B)) = \text{rank}(B) \leq \min(d, n) \leq d,
\]
and it was established before that \( d \) is \( O\left(\frac{\log(n)}{\varepsilon^2}\right) \).