Exercise 1 (Distance between $\ell_1$ and $\ell_\infty$)
Construct a linear map $T : \ell^1_n \to \ell^\infty_n$ satisfying $\|T\|_{op} \|T^{-1}\|_{op} = O(\sqrt{n})$. The goal of this exercise is to extend the Hadamard based construction in Lecture 1 from $n$ a power of 2 to all $n$. (Hint: Use the binary expansion of $n$.)

Exercise 2 (Little Grothendieck Inequality)
In this exercise you will show that for positive semidefinite matrices, the Grothendieck constant is at most $\pi/2$. Recall that a matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (PSD) if and only if it can be written as $A = B^T B$ for some (not necessarily square) matrix $B$.

1. Show that if $A \in \mathbb{R}^{n \times n}$ is PSD, then
$$\max_{x, y \in S_2^{n-1}} \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle = \max_{x \in S_2^n} \sum_{i,j=1}^n A_{ij} \langle x_i, x_j \rangle.$$ 

2. Show that for any $d \in \mathbb{N}$ and vectors $x_1, \ldots, x_n \in S_2^{d-1}$, the matrix $B \in \mathbb{R}^{n \times n}$ given by
$$B_{ij} = \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle) - \frac{2}{\pi} \langle x_i, x_j \rangle$$
is PSD.
(Hint: Use the Taylor expansion of the arcsin function:
$$\arcsin(t) = \sum_{k=0}^\infty a_k t^{2k+1} \quad \forall t \in [-1, 1], \quad a_k = \frac{(2k)!}{4^k (k!)^2 (2k + 1)^k},$$
and that $\langle x, y \rangle^k = \langle x^{\otimes k}, y^{\otimes k} \rangle$.)

3. Conclude that for any $n \in \mathbb{N}$ and PSD matrix $A \in \mathbb{R}^{n \times n}$, we have $\|A\|_{C} \leq (\pi/2) \|A\|_{\ell_\infty \to \ell_1}$. (Hint: Use Grothendieck’s identity and that if $B, C \in \mathbb{R}^{n \times n}$ are PSD, then $\sum_{i,j} B_{ij} C_{ij} \geq 0$.)

Exercise 3 (Covering with an Asymmetric Convex Body)
Let $B \subseteq \mathbb{R}^n$ be a convex body. In class, we gave volumetric estimates on covering numbers when $B$ is symmetric, which you will extend to the asymmetric case here.

1. Show that $E[\text{vol}_n(B \cap (2x - B))/\text{vol}_n(B)] = 1/2^n$, where $x$ is sampled uniformly from $B$.
(Hint: Show that the above expectation can be expressed as $\text{vol}_n(B)^{-2} \int_B \int_B 1[y \in (2x - B)]dydx$ and exchange the order of integration.)

2. Use the above to show that there exists a symmetric convex body $K \subseteq \mathbb{R}^n$ and a shift $t \in \mathbb{R}^n$ such that $K + t \subseteq B$ and $\text{vol}_n(K) \geq 2^{-n} \text{vol}_n(B)$.
(Hint: Note that $2x - B$ is the reflection of $B$ about $x$.)

3. For $K$ as above and $A \subseteq \mathbb{R}^n$, show that
$$N(A, B) \leq N(A, K) \leq 4^n \text{vol}_n(A + B/2)/\text{vol}_n(B).$$
In particular, show that $N(B, K) \leq 6^n$ and $N(B - B, B) \leq 60^n$.
(Hint: Use the packing argument with $B$ replaced by $K$ and compare the bounds.)

Exercise 4 (Low Rank Approximation of the Identity)
Let $I_n \in \mathbb{R}^{n \times n}$ denote the $n \times n$ identity matrix. For $\varepsilon \in (0, 1)$, show that there exists a positive semidefinite matrix $\tilde{I}_n \in \mathbb{R}^{n \times n}$ such that $|(I_n - \tilde{I}_n)_{ij}| \leq \varepsilon, \forall i, j \in [n]$ and $\text{rank} (\tilde{I}_n) = O(\log n/\varepsilon^2)$.
(Hint: Apply the Johnson-Lindenstrauss lemma to the canonical rank factorization of $I_n$.)