Exercise 1 (Distance between ℓ_1 and ℓ_∞)
Construct a linear map $T : \ell_1^n \to \ell_\infty^n$ satisfying $\|T\|_{\mathrm{op}}\|T^{-1}\|_{\mathrm{op}} = O(\sqrt{n})$. The goal of this exercise is to extend the Hadamard based construction in Lecture 1 from n a power of 2 to all n. (Hint: Use the binary expansion of n.)

Exercise 2 (Little Grothendieck Inequality)
In this exercise you will show that for positive semidefinite matrices, the Grothendieck constant is at most $\pi/2$. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite (PSD) if and only if it can be written as $A = B^\mathsf{T} B$ for some (not necessarily square) matrix B.

1. Show that if $A \in \mathbb{R}^{n\times n}$ is PSD, then
$$\max_{x_i, y_i \in S_2^{2n-1}} \sum_{i,j=1}^n A_{ij}\langle x_i, y_j\rangle = \max_{x_i \in S_2^{n-1}} \sum_{i,j=1}^n A_{ij}\langle x_i, x_j\rangle.$$
2. Show that for any $d \in \mathbb{N}$ and vectors $x_1, \ldots, x_n \in S_2^{d-1}$, the matrix $B \in \mathbb{R}^{n\times n}$ given by
$$B_{ij} = \frac{2}{\pi}\arcsin(\langle x_i, x_j\rangle) - \frac{2}{\pi}\langle x_i, x_j\rangle$$
is PSD.
(Hint: Use the Taylor expansion of the arcsin function:
$$\arcsin(t) = \sum_{k=0}^\infty a_k t^{2k+1} \qquad \forall t \in [-1,1], \qquad a_k = \frac{(2k)!}{4^k (k!)^2 (2k+1)},$$
and that $\langle x, y\rangle^k = \langle x^{\otimes k}, y^{\otimes k}\rangle$.)
3. Conclude that for any $n \in \mathbb{N}$ and PSD matrix $A \in \mathbb{R}^{n\times n}$, we have $\|A\|_G \le (\pi/2)\|A\|_{\ell_\infty \to \ell_1}$.
(Hint: Use Grothendieck's identity and that if $B, C \in \mathbb{R}^{n\times n}$ are PSD, then $\sum_{i,j} B_{ij}C_{ij} \ge 0$.)

Exercise 3 (Covering with an Asymmetric Convex Body)
Let $B \subseteq \mathbb{R}^n$ be a convex body. In class, we gave volumetric estimates on covering numbers when B is symmetric, which you will extend to the asymmetric case here.

1. Show that $\mathbb{E}[\mathrm{vol}_n(B \cap (2x - B))/\mathrm{vol}_n(B)] = 1/2^n$, where x is sampled uniformly from B.
(Hint: Show that the above expectation can be expressed as $\mathrm{vol}_n(B)^{-2}\int_B\int_B 1[y \in (2x - B)]dydx$ and exchange the order of integration.)
2. Use the above to show that there exists a symmetric convex body $K \subseteq \mathbb{R}^n$ and a shift $t \in \mathbb{R}^n$ such that $K + t \subseteq B$ and $\mathrm{vol}_n(K) \ge 2^{-n}\mathrm{vol}_n(B)$.
(Hint: Note that $2x - B$ is the reflection of B about x.)
3. For K as above and $A \subseteq \mathbb{R}^n$, show that
$$N(A, B) \le N(A, K) \le 4^n \mathrm{vol}_n(A + B/2)/\mathrm{vol}_n(B)\ .$$
In particular, show that $N(B, K) \le 6^n$ and $N(B - B, B) \le 60^n$.
(Hint: Use the packing argument with B replaced by K and compare the bounds.)

Exercise 4 (Low Rank Approximation of the Identity) Let $I_n \in \mathbb{R}^{n\times n}$ denote the $n \times n$ identity matrix. For $\varepsilon \in (0, 1)$, show that there exists a positive semidefinte matrix $\tilde{I}_n \in \mathbb{R}^{n\times n}$ such that $|(I_n - \tilde{I}_n)_{ij}| \le \varepsilon$, $\forall i, j \in [n]$ and $\mathrm{rank}(\tilde{I}_n) = O(\log n/\varepsilon^2)$.
(Hint: Apply the Johnson-Lindenstrauss lemma to the canonical rank factorization of I_n.)