Geometry and Distances of ℓ_p^n Spaces

In these notes, we will give good estimates of the "size" of the ℓ_p^n unit balls and tight bounds on the distances between ℓ_p^n spaces.

Comparing ℓ_p norms. We begin with the basic comparison inequalities for the ℓ_p norms on $\mathbb{R}^n$.

LEMMA 1 *For $1 \leq p < q \leq \infty$, $x \in \mathbb{R}^n$, the following holds:*

$$\|x\|_q \leq \|x\|_p \leq n^{1/p-1/q}\|x\|_q.$$

Furthermore, the equality cases satisfy:

1. $\|x\|_q = \|x\|_p$ *iff x has at most one non-zero coordinate.*
2. $\|x\|_p = n^{1/p-1/q}\|x\|_q$ *iff* $|x_1| = \cdots = |x_n|$.

PROOF: We prove the above for $1 < p < q < \infty$, leaving the remaining cases ($p = 1$ or $q = \infty$) as a simple exercise. For the first inequality, we see that

$$\begin{aligned}\|x\|_q = (\sum_{i=1}^n |x_i|^q)^{1/q} &\leq (\max_{i\in[n]} |x_i|^{q-p} \sum_{i=1}^n |x_i|^p)^{1/q} = (\max_{i\in[n]} |x_i|)^{(q-p)/q} \cdot \|x\|_p^{p/q} \\ &\leq ((\sum_{i\in[n]} |x_i|^p)^{1/p})^{(q-p)/q} \cdot \|x\|_p^{p/q} = \|x\|_p^{(q-p)/q} \cdot \|x\|_p^{p/q} = \|x\|_p.\end{aligned}$$

For the equality condition here, note that the second inequality can only be tight if $\max_{i\in[n]} |x_i|^p = \sum_{i\in[n]} |x_i|^p$, which is clearly iff x at most one non-zero coordinate. Since the first inequality is also tight if x has at most non-zero coordinate (note we also have tightness if all the non-zero coordinates have the same absolute value in this setting), this fully describes the equality condition.

For the second inequality, we apply Holder's inequality with $1/r = p/q$ and $1/s = (q-p)/q$, to get

$$\|x\|_p = (\sum_{i=1}^n |x_i|^p \cdot 1)^{1/p} \leq ((\sum_{i=1}^n |x_i|^{pr})^{1/r}(\sum_{i=1}^n 1^s)^{1/s}))^{1/p} = (\sum_{i=1}^n |x_i|^q)^{1/q} \cdot n^{(q-p)/(pq)} = \|x\|_q \cdot n^{1/p-1/q}.$$

Recall that equality for Holder's inequality holds above iff for some constant $c \geq 0$ we have $|x_i|^q = c \cdot 1$, $\forall i \in [n]$. Equivalently iff $|x_1| = \cdots = |x_n|$ as needed. □

As a direct corollary to the above, we get the following inclusions and distance upper bounds:

COROLLARY 2 *For $1 \leq p \leq q \leq \infty$, the following holds:*

1. $B_p^n \subseteq B_q^n \subseteq n^{1/p-1/q} B_p^n$.
2. $d(\ell_p^n, \ell_q^n) \leq n^{1/q-1/q}$.

PROOF: For part 1, note that the first inclusion $B_p^n \subseteq B_q^n \Leftrightarrow \|x\|_q \leq \|x\|_p$, $\forall x \in \mathbb{R}^n$ and the second $B_q^n \subseteq n^{1/p-1/q} B_p^n \Leftrightarrow \|x\|_p \leq n^{1/p-1/q}\|x\|_q$, $\forall x \in \mathbb{R}^n$. As these inequalities are exactly those in 1, the statement follows.

For part 2, we examine the identity map I_n on $\mathbb{R}^n$ as a map from ℓ_p^n to ℓ_q^n. To upper bound $d(\ell_p, \ell_q)$ by $n^{1/p-1/q}$, it suffices to show that $\|I_n\|_{p\to q}\|I_n\|_{q\to p} \leq n^{1/p-1/q}$. From here, we have that bounds $\|I_n\|_{p\to q} \leq 1$ since $\|x\|_q \leq \|x\|_p$ and $\|I_n\|_{q\to p} \leq n^{1/p-1/q}$ since $\|x\|_p \leq n^{1/p-1/q}\|x\|_q$, as needed. □

Volumes of B_p^n balls. We will now compute useful bounds on the volumes of the B_p^n balls. Since for any measurable set $A \subseteq \mathbb{R}^n$, we have $\mathrm{vol}_n(tA) = t^n \mathrm{vol}_n(A)$ (i.e. n-homogeneity of volume) for $t \geq 0$, a reasonable measure of the "size" of A is its normalized volume $\mathrm{vol}_n(A)^{1/n}$. Somewhat surprisingly, we will be able to estimate these normalized volumes of all B_p^n balls up to a constant factor by simply comparing the extreme cases, the octahedron B_1^n and cube $B_\infty^n = [-1,1]^n$.

Before starting these computations, we first give the n-dimensional analog of the "base" $\times$ "height" volume in $\mathbb{R}^2$.

Lemma 3 *Let $A \subseteq \mathbb{R}^{n-1}$ be measurable and $t \geq 0$. Then*

$$\mathrm{vol}_n(\{(x,s) : x \in (s/t)A, s \in [0,t]\}) = \frac{t}{n}\mathrm{vol}_{n-1}(A).$$

Furthermore, if A is convex then the convex hull

$$\mathrm{conv}((A,0) \cup \{(0,t)\}) = \{(x,s) : x \in (s/t)A, s \in [0,t]\}.$$

Proof: For the first part, by Fubini

$$\begin{aligned}
\mathrm{vol}_n(\{(x,s) : x \in (s/t)A, s \in [0,t]\}) &= \int_0^t \int_{\mathbb{R}^{n-1}} 1_{x \in (s/t) \in A} dx ds \\
&= \int_0^t \mathrm{vol}_{n-1}((s/t)A) ds = \mathrm{vol}_{n-1}(A) \int_0^t (s/t)^{n-1} ds \\
&= \mathrm{vol}_{n-1}(A) \cdot t \int_0^1 s^{n-1} ds = \mathrm{vol}_{n-1}(A) \cdot t/n,
\end{aligned}$$

as needed. We leave the furthermore as an exercise to the reader. □

We give our estimates for the volume of the ℓ_p^n balls below:

Lemma 4 *For $n \geq 1$, the following holds:*

1. $\mathrm{vol}_n(B_\infty^n)^{1/n} = 2$.
2. $\mathrm{vol}_n(B_1^n)^{1/n} = 2/(n!)^{1/n} \leq 2e/n$.
3. *For $p \in (1,\infty)$, we have*

$$2n^{-1/p} = \mathrm{vol}_n(n^{-1/q} B_\infty^n)^{1/n} \leq \mathrm{vol}_n(B_p^n)^{1/n} \leq \mathrm{vol}_n(n^{1-1/q} B_1^n) \leq 2en^{-1/p}.$$

Proof: For part 1, we have that $\mathrm{vol}_n(B_\infty^n)^{1/n} = \mathrm{vol}_n([-1,1]^n)^{1/n} = \mathrm{vol}_1([-1,1]) = 2$.

For part 2, let $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^n x_i \leq 1\}$. By symmetry,

$$\begin{aligned}
\mathrm{vol}_n(B_1^n) &= \mathrm{vol}_n(\{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}) = \mathrm{vol}_n(\cup_{s \in \{-1,1\}^n} \{(x_1 s_1, \ldots, x_n s_n) \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^n x_i \leq 1\}) \\
&= \sum_{s \in \{-1,1\}^n} \mathrm{vol}_n(\{(x_1 s_1, \ldots, x_n s_n) \in \mathbb{R}^n : x \in \Delta_n\}) = 2^n \mathrm{vol}_n(\Delta_n).
\end{aligned}$$

Above, the second to last equality holds since the intersections have measure zero and the last equality holds since the linear map $(x_1, \ldots, x_n) \to (x_1 s_1, \ldots, s_n x_n)$ is measure preserving for $s \in \{-1, 1\}^n$.

From here, for $n = 1$, note that $\Delta_1 = [0, 1]$ and hence $\mathrm{vol}_1(\Delta_1) = 1$. For $n \geq 2$, it is direct to verify that $\Delta_n = \mathrm{conv}((\Delta_{n-1}, 0) \cup \{(0, 0)\})$. Therefore, by Lemma 3, we get that $\mathrm{vol}_n(\Delta_n) = \mathrm{vol}_{n-1}(\Delta_{n-1})/n$. Thus, by induction, $\mathrm{vol}_n(\Delta_n) = 1/(n!)$. This yields the exact formula $\mathrm{vol}_n(B_1^n)^{1/n} = 2\mathrm{vol}_n(\Delta_n)^{1/n} = 2/(n!)^{1/n}$. Using the inequality $n^n/(n!) \leq e^n$ (i.e. expand the Taylor series), we conclude that $\mathrm{vol}_n(B_1^n)^{1/n} \leq 2e/n$, as desired.

For part 3, it follows directly the parts 1 and 2 combined with the inclusions in Corollary 2 part 1, namely $n^{-1/p} B_\infty^n \subseteq B_p^n \subseteq n^{1-1/p} B_1^n$. $\square$

We note that it is perhaps surprising that the most "obvious" sandwiching $B_1^n \subseteq B_p^n \subseteq B_\infty^n$ yields lower and upper normalized volume estimates that are a $\Theta(n)$ factor off from each other. The fact that the "reverse" inclusions are tighter suggests that most of the volume of the cube B_∞^n is close to its vertices while most of the volume of B_1^n is close to the center of its facets. We note that computing an exact expression for the volume of the ℓ_p^n balls can be done relatively simply and will be the subject of one of the exercises.

Distances between ℓ_p^n spaces. We will now show that one can get constant factor tight estimates for the Banach-Mazur distance between ℓ_p^n and ℓ_q^n for all ranges of p and q. In general, it is extremely difficult to compute good estimates for the Banach-Mazur distance between normed spaces. In particular, lower bounds must hold against any possible way of mapping one space bijectively into the other. The ℓ_p^n spaces are indeed one of the very few classes spaces for which these distances are known and thus give nice examples for how such bounds can be proved.

The general formula for the distance between ℓ_p^n spaces is given below.

Theorem 5 (ℓ_p^n distances) *For $n \in \mathbb{N}$, $1 \leq p \leq q \leq \infty$, the following holds:*

$$d(\ell_p, \ell_q) = \begin{cases} n^{1/p-1/q}: & p \leq q \leq 2 \text{ or } 2 \leq p \leq q \\ \Theta(\max\{n^{1/p-1/2}, n^{1/2-1/q}\}): & p \leq 2 \leq q \end{cases}. \tag{1}$$

The formulas above are self-dual, i.e. they are invariant under $(p, q) \to (q/(q-1), p/(p-1))$ (note the order switches as we ask for $p \geq q$ for the formulas to be valid), which one would expect since Banach-Mazur distance is invariant under duality in finite dimensions. The precise distances are known only for the range $2 \leq p \leq q$ and $p \leq q \leq 2$, where it will turn out that the identity map is optimal. For the range $p \leq 2 \leq q$, the formula suggests that $d(\ell_p^n, \ell_q^n) = \Theta(1) \max\{d(\ell_p, \ell_{p/p-1}), d(\ell_q^n, \ell_{q/(q-1)})\}$, is the maximum distance between ℓ_p^n, ℓ_q^n and their respective duals. Thus in this range, we are in fact measuring the distance to one's dual. In this setting, the optimal map will no longer be the identity and instead will be a certain Hadamard like orthogonal transformation.

An important point given by the formulas is that the maximum distance between any ℓ_p^n and ℓ_q^n is in fact $\Theta(\sqrt{n})$, which is achieved only for the settings $(p, q) \in \{(1, 2), (2, \infty), (1, \infty)\}$. As we will see in the next lectures, in the worst-case, any two n-dimensional normed spaces are at distance at most n. In fact, we will show that any n-dimensional normed space is at distance at most $\sqrt{n}$ from ℓ_2^n, which we see is tight from ℓ_2^n vs ℓ_1^n or ℓ_∞^n. For a detailed accounting of the subject of Banach-Mazur distances, we encourage the reader to consult the monograph of Tomczak-Jaegermann [TJ89].

Interestingly, all of the above bounds are essentially achieved by "interpolating" through the extreme cases $(p,q) \in \{(1,2),(2,\infty),(1,\infty)\}$. The arguments are mostly elementary with the exception of proving upper bounds on the distances for the ranges $p \leq 2 \leq q$, where we will need non-obvious upper bounds on the p to q operator norm of a Hadamard like transformation. For this purpose, we will rely on the Riesz-Thorin interpolation theorem, which we state below:

THEOREM 6 (RIESZ-THORIN INTERPOLATION) *Let $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Let $p_1, p_2, q_1, q_2 \in [1,\infty]$, $\theta \in (0,1)$. Define $p_\theta, q_\theta \in [2,\infty]$ to be*

$$p_\theta = (\theta/p_1 + (1-\theta)/p_2)^{-1},$$
$$q_\theta = (\theta/q_1 + (1-\theta)/q_2)^{-1},$$

the corresponding harmonic averages of p_1, p_2 and q_1, q_2. Then

$$\|T\|_{p_\theta \to q_\theta} \leq \|T\|_{p_1 \to q_1}^{\theta} \|T\|_{p_2 \to q_2}^{1-\theta}.$$

The above theorem allows us to get upper bounds on operator norms that are "in between" bounds we already know. The proof relies on the concept of complex interpolation of Banach spaces and Hadamard's three lines lemma, which we may cover towards the end of the course. We content ourselves for now on using this theorem as a black-box, as it will allow us to get the full picture for distances between ℓ_p^n spaces.

We now prove distance bounds for the base cases $p, q \in \{1, 2, \infty\}$.

LEMMA 7 *For $n \in N$, the following holds:*

1. $d(\ell_1^n, \ell_2^n) = d(\ell_2^n, \ell_\infty^n) = \sqrt{n}$.
2. $d(\ell_1^n, \ell_\infty^n) = \Theta(\sqrt{n})$.

PROOF: For part 1, by duality it suffices to prove the bound for ℓ_2^n and ℓ_∞^n. By corollary 2, we have that $d(\ell_2^n, \ell_\infty^n) \leq \sqrt{n}$. Thus it suffices to prove the lower bound. Let $T \in \mathcal{L}(\mathbb{R}^n)$ be invertible. By scaling, we may assume that $\|T\|_{2\to\infty} = 1$ and thus must prove that $\|T^{-1}\|_{\infty\to 2} \geq \sqrt{n}$. To prove the lower bound on the operator, for the standard basis $e_1, \ldots, e_n \in \mathbb{R}^n$, note that $1 = \|e_i\|_\infty \leq \|T\|_{2\to\infty}\|T^{-1}e_i\|_2 = \|T^{-1}e_i\|_2$.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be uniform $\{-1,1\}$ random variables. From here, by the parallelogram law, we have that

$$\mathbb{E}[\|T^{-1}(\sum_{i=1}^n \varepsilon_i e_i)\|_2^2] = \mathbb{E}[\sum_{i,j\in[n]} \varepsilon_i\varepsilon_j \langle T^{-1}e_i, T^{-1}e_j\rangle] = \sum_{i=1}^n \langle T^{-1}e_i, T^{-1}e_i\rangle = \sum_{i=1}^n \|T^{-1}e_i\|_2^2 \geq n.$$

The second to last equality above follows since $\varepsilon_i^2 = 1, \ \forall i \in [n]$, and $\mathbb{E}[\varepsilon_i\varepsilon_j] = \mathbb{E}[\varepsilon_i]\mathbb{E}[\varepsilon_j] = 0$ for $i \neq j$. Given the above, by averaging, there must exist signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}$ such that $\|T^{-1}(\sum_{i=1}^n \varepsilon_i e_i)\|_2 \geq \sqrt{n}$. Since $\|\sum_{i=1}^n \varepsilon_i e_i\|_\infty = 1$, this proves that $\|T^{-1}\|_{\infty\to 2} \geq \sqrt{n}$, as needed.

We now prove part 2. We begin with the lower bound, which will use volumetric information. Let $T \in \mathcal{L}(\mathbb{R}^n)$ be invertible. Again by scaling, we may assume that $\|T\|_{1\to\infty} = 1$ and thus we must show that $\|T^{-1}\|_{\infty\to 1} = \Omega(\sqrt{n})$. By the first condition, we see that $\|Te_i\|_2 \leq \sqrt{n}\|Te_i\|_\infty \leq \sqrt{n}\|T\|_{1\to\infty}\|e_i\|_1 \leq \sqrt{n}$. By Hadarmard's inequality, we in fact have that

$$|\det(T)|^{1/n} \leq (\prod_{i=1}^n \|Te_i\|_2)^{1/n} \leq \sqrt{n}$$

Recalling that $B_\infty^n \subseteq \|T^{-1}\|_{\infty\to 1} TB_1^n$, we must have

$$\begin{aligned} \mathrm{vol}_n(B_\infty^n)^{1/n} &\leq \mathrm{vol}_n(\|T^{-1}\|_{\infty\to 1}(TB_1^n))^{1/n} \Leftrightarrow \\ \frac{\mathrm{vol}_n(B_\infty^n)^{1/n}}{|\det(T)|^{1/n}\mathrm{vol}_n(B_1^n)^{1/n}} &\leq \|T^{-1}\|_{\infty\to 1} \Rightarrow \\ \sqrt{n}/e &\leq \|T^{-1}\|_{\infty\to 1}, \end{aligned}$$

as needed.

For the upper bound, we prove it when n is a power of 2, leaving the general case as an exercise. For n a power of 2, we will map ℓ_1^n to ℓ_∞^n using the so-called Hadamard transform H_n. For each $i \in [n]$, let $b(i) \in \{0,1\}^{\log_2 n}$ denote the binary expansion of $i-1$, i.e. $i-1 = \sum_{j=0}^{\log_2 n - 1} 2^j b_{j+1}$. Note that $b : [n] \to \{0,1\}^{\log_2 n}$ is clearly bijective when n is a power of 2. With this indexing, we may express the coefficient matrix of H_n by $(H_n)_{ij} = (-1)^{\langle b(i), b(j)\rangle}$, $\forall i,j \in [n]$. It is an exercise to check that the columns of H_n are orthogonal and all have ℓ_2 norm $\sqrt{n}$. In particular, we get that $\|H_n x\|_2 = \sqrt{n}\|x\|_2 \Leftrightarrow \|H_n^{-1}x\|_2 = \|x\|_2/\sqrt{n}$ for all $x \in \mathbb{R}^n$. We now show that $\|H_n\|_{1\to\infty} \leq 1$ and $\|H_n^{-1}\|_{\infty\to 1} \leq \sqrt{n}$. The first inequality is direct since the columns of H_n have entries in ± 1. The second inequality is inequality is derived by comparing to ℓ_2:

$$\|H_n^{-1}y\|_1 \leq \sqrt{n}\|H_n^{-1}y\|_2 = \|y\|_2 \leq \sqrt{n}\|y\|_\infty, \ \forall y \in \mathbb{R}^n.$$

Thus, $\|H_n^{-1}\|_{\infty\to 1} \leq \sqrt{n}$ as needed. □

We now use the above to prove Theorem 5.

Proof:[Proof of Theorem 5] We start with the range $p \leq q \leq 2$ or $2 \leq p \leq q$. By duality, we may restrict to range $2 \leq p \leq q$. Applying Lemma 7 part 1 and Corollary 2 part 2 together with submultiplicativy of Banach-Mazur distance, we see that

$$\sqrt{n} = d(\ell_2^n, \ell_\infty^n) \leq d(\ell_2^n, \ell_p^n) d(\ell_p^n, \ell_q^n) d(\ell_q^n, \ell_\infty^n) \leq n^{1/2-1/p} n^{1/p-1/q} n^{1/q} = \sqrt{n}.$$

Given that the left hand side and right hand side are equal, we must have equality throughout. In particular, we get $d(\ell_p^n, \ell_q^n) = n^{1/p-1/q}$.

We move to the range $p \leq 2 \leq q$. By duality, we may assume that $1/p - 1/2 \geq 1/2 - 1/q$. In contrast to the previous case, we will note be able to use the estimate $d(\ell_1, \ell_\infty) = \Theta(\sqrt{n})$ in a black-box manner, however the proof will be almost identical to the case of ℓ_1 vs ℓ_∞.

We begin with the lower bound. Let $T \in \mathcal{L}(\mathbb{R}^n)$ be invertible and assume that $\|T\|_{p\to q} = 1$. Let $d = \|T^{-1}\|_{q\to p}$. As before, we must prove that $d \geq \Omega(n^{1/p-1/2})$. Note that for $i \in [n]$,

$$\|Te_i\|_2 \leq n^{1/2-1/q}\|Te_i\|_q \leq n^{1/2-1/q}\|T\|_{p\to q}\|e_i\|_q = n^{1/2-1/q},$$

where the last equality holds by assumption on T. Therefore, by Hadamard's inequality $|\det(T)|^{1/n} \leq n^{1/p-1/2}$. Since $B_q^n \subseteq dTB_p^n$, using the volume bounds in Lemma 4, we get as before that

$$d \geq \frac{\mathrm{vol}_n(B_q^n)^{1/n}}{|\det(T)|^{1/n}\mathrm{vol}_n(B_p^n)^{1/n}} \geq n^{1/p-1/2}/e,$$

as needed.

For the upper bound, we assume as before that n is a power of 2, leaving the general case as an exercise. We use the Hadamard transformation H_n as before. Recall that $\|H\|_{2\to 2} = \sqrt{n}$

and $\|H\|_{1\to\infty} = 1$. Let $\theta = 2/q$ where $p_\theta = (\theta \cdot 1/2 + (1-\theta)\cdot 1)^{-1} = q/(q-1)$ and $q_\theta = (\theta \cdot 1/2 + (1-\theta)\cdot 1/\infty) = q$. Note that $1/p_\theta + 1/q = 1$, so these form a dual pair. Furthermore, our assumption that $1/p + 1/q \geq 1 \Rightarrow p_\theta = q/(q-1) \geq p$. Since $\|x\|_{p_\theta} \leq \|x\|_p$, $\forall x \in \mathbb{R}^n$, we see that

$$\|H_n\|_{p\to q} \leq \|H_n\|_{p_\theta \to q} = \|H_n\|_{p_\theta \to q_\theta}.$$

From here, applying Riesz-Thorin interpolation (Theorem 6), we get that

$$\|H_n\|_{p_\theta \to q_\theta} \leq \|H_n\|_{2\to 2}^{\theta} \|H_n\|_{1\to\infty}^{1-\theta} = n^{1/q}.$$

Thus $\|H_n\|_{p\to q} \leq n^{1/q}$. For the bound on $\|H_n^{-1}\|_{q\to p}$, we again compare directly to ℓ_2, which gives

$$\|H_n^{-1}y\|_p \leq n^{1/p-1/2}\|H_n^{-1}y\|_2 = n^{1/p-1}\|y\|_2 \leq n^{1/p-1}(n^{1/2-1/q}\|y\|_q) = n^{1/p-1/2-1/q}\|y\|_q, \forall y \in \mathbb{R}^n.$$

Thus, $\|H^{-1}\|_{q\to p} \leq n^{1/p-1/2-1/q}$. This yields the final distortion bound

$$\|H_n\|_{p\to q}\|H_n^{-1}\|_{q\to p} \leq n^{1/q} \cdot n^{1/p-1/2-1/q} = n^{1/p-1/2},$$

as needed. □

References

[TJ89] Nicole Tomczak-Jaegermann. *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, volume 38 of *Pitman monographs and surveys in pure and applied Mathematics*. Longman Scientific & Technical, 1989. Available at `https://sites.ualberta.ca/~ntj/bm_book/NTJ_BM_distance.djvu`.