GEOMETRIC FUNCTIONAL ANALYSIS AND APPLICATIONS

—LECTURE NOTES— GROTHENDIECK'S INEQUALITY

These notes are provided for your convenience and may contain errors. If you find an error, please leave a message on the student forum.

1. Grothendieck's inequality

This lecture is about Grothendieck's inequality, the centerpiece of the extraordinary paper *"Résumé de la théorie métrique des produits tensoriels topologiques"* [Gro53]. This result shows a surprising relation between the three fundamental Banach spaces ℓ_1, ℓ_2 and ℓ_∞. Denote by $S_p^{n-1} = \{x \in \mathbb{R}^n : \|x\|_p = 1\}$ the unit sphere of $\ell_p^n = (\mathbb{R}^n, \|\,\|_p)$. The simplest formulation of Grothendieck's inequality is given in terms of the following two quantities on $\mathbb{R}^{n\times n}$:

$$\|A\|_{\ell_\infty \to \ell_1} = \sup\Big\{ \sum_{i,j=1}^n A_{ij} a_i b_j : a, b \in B_\infty^n \Big\}$$

$$\|A\|_G = \sup\Big\{ \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle : d \in \mathbb{N},\, x_i, y_j \in B_2^d \Big\}.$$

The notation suggests that these quantities are norms, which they are. The first is easily seen to be the operator norm of the linear operator from ℓ_∞^n to ℓ_1^n given by $x \mapsto Ax$ (hence the notation). We leave showing that the second is also a norm as an exercise. Let us make a few preliminary observations of these norms. First, by convexity and compactness of B_p^n and bi-linearity of the arguments, the suprema are attained by vectors $a, b \in \{-1, 1\}^n$ and $x_i, y_j \in S_2^{2n-1}$, respectively, where the second fact follows because there are only $2n$ vectors appearing in $\|A\|_G$ (spanning a vector space of dimension at most $2n$). Second, $\|A\|_G \geq \|A\|_{\ell_\infty\to\ell_1}$ holds for any matrix A, since $B_2^1 = [-1, 1]$. Third, the last inequality can be strict, as can be seen from the Hadamard matrix $H = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. On the one hand, $\|H\|_{\ell_\infty\to\ell_1} = 2$ (exercise).

Date: March 12, 2019.

On the other, $\|H\|_G \geq 2\sqrt{2}$, which can be seen by considering the 2-dimensional unit vectors shown in Figure 1.

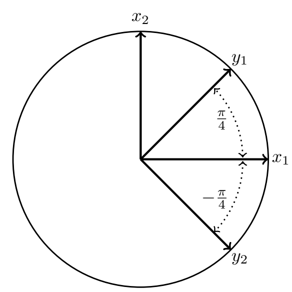

Figure 1. Vectors for Hadamard matrix.

Surprisingly, Grothendieck's inequality shows that these norms are never too far apart, however.

Theorem 1.1 (Grothendieck's inequality)**.** *There exists an absolute constant $K \in (1, \infty)$ such that the following holds. For any positive integer n and matrix $A \in \mathbb{R}^{n \times n}$, we have*

$$\|A\|_{\ell_\infty \to \ell_1} \leq \|A\|_G \leq K\|A\|_{\ell_\infty \to \ell_1}. \tag{1}$$

There are many equivalent formulations of this result. Its original formulation in [Gro53] was in terms of norms on tensor products of Banach spaces. The form used above is due to Lindenstrauss and Pełczyński [LP68], who revamped Grothendieck's original work in such a way so as to lift it from an obscurity it had unfortunately suffered up until then. We refer to Pisier's survey [Pis12] for more information about its interesting history and ramifications. The *Grothendieck constant* K_G is the smallest K for which Theorem 1.1 holds true. Determining its exact value is the only one of six problems posed in [Gro53] that remains open to this day. The Hadamard matrix shows that $K_G \geq \sqrt{2}$. The best bounds $1.6769\cdots \leq K_G < 1.7822\ldots$ were proved by Davie and Reeds [Dav84, Ree91], and Braverman et al. [BMMN13], respectively. In the next section, we give arguably the most elegant proof of Theorem 1.1, due to Krivine [Kri79], who showed that

$$K_G \leq \frac{\pi}{2\ln(1+\sqrt{2})} = 1.7822\ldots.$$

The elegance of Krivine's proof led many researchers to believe that this was in fact the exact value of K_G. No one could prove this, however, and it turns out for the good reason that it is false. It was shown relatively recently in [BMMN13] that K_G is strictly smaller than Krivine's bound by some additive $\varepsilon > 0$. Unfortunately, the proof of this fact is based on a long series of calculations in complex analysis in addition to a computer-assisted search for a good partition of the plane into two disjoints sets (giving the so-called tiger partition shown in Figure 2).

Figure 2. The tiger partition. Source: `https://web.math.princeton.edu/~naor/`

2. Krivine's proof of Grothendieck's inequality

The first ingredient of Krivine's proof of Theorem 1.1 is the following simple lemma, which was also used in the original proof given in [Gro53], but in a less effective way (giving a larger value of K).

Lemma 2.1 (Grothendieck's identity)**.** *Let x, y be n-dimensional real unit vectors and let $g = (g_1, \dots, g_n) \sim N(0, I_n)$ be an n-dimensional standard Gaussian vector. Then,*

$$\mathbb{E}\big[\operatorname{sign}(\langle x, g\rangle)\operatorname{sign}(\langle y, g\rangle)\big] = \frac{2}{\pi}\arcsin(\langle x, y\rangle). \tag{2}$$

Proof sketch: If $x = y$ or $x = -y$ then the identity is trivial. Suppose that x and y are not parallel and consider the two-dimensional subspace spanned by them. By rotational invariance of the Gaussian distribution, the projection of g onto this subspace is a two-dimensional standard Gaussian. Observe that $\operatorname{sign}(\langle x, g\rangle)\operatorname{sign}(\langle y, g\rangle)$ is positive if and only if g lies above or below both of the half-planes orthogonal to x and y respectively (Figure 3).

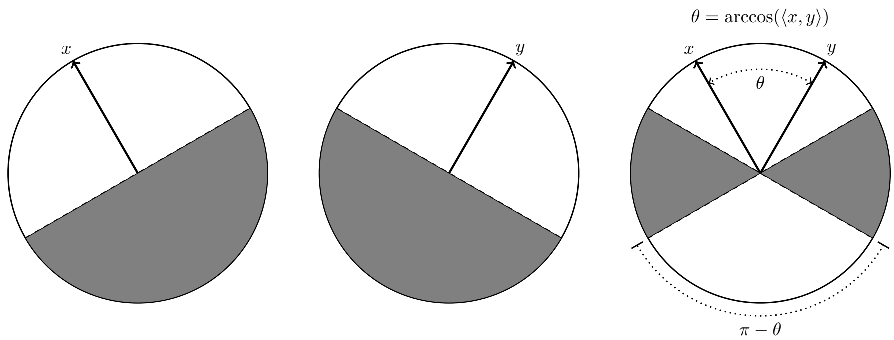

Figure 3. Grothendieck's identity in two dimensions.

Since the direction of g is uniform on the unit circle, it follows that this happens with probability

$$\frac{2}{2\pi}\big(\pi - \arccos(\langle x, y\rangle).$$

Hence, the expectation in (2) equals

$$\frac{1}{\pi}\big(\pi - \arccos(\langle x, y\rangle) - \frac{1}{\pi}\big(\arccos(\langle x, y\rangle) = 1 - \frac{2}{\pi}\arccos(\langle x, y\rangle),$$

which equals the right-hand side of (2). □

We will use the Taylor expansions of the sine and hyperbolic sine functions, given by

$$\sin(t) = \sum_{k=0}^{\infty} \alpha_k t^k$$

$$\sinh(t) = \sum_{k=0}^{\infty} |\alpha_k| t^k,$$

where for $k = 0, 1, \ldots,$ we have $\alpha_{2k} = 0$ and $\alpha_{2k+1} = (-1)^k/((2k+1)!)$. In particular, the Taylor coefficients of the hyperbolic sine functions are given by the absolute values of those of the sine function. Moreover, both of these Taylor series converge on the interval $[-1, 1]$.

Recall that the *tensor product* of vectors $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_1}$ is the $d_1 d_2$-dimensional vector $x \otimes y \in \mathbb{R}^{d_1 d_2}$ given by

$$x \otimes y = (x_i y_j)_{(i,j) \in [d_1] \times [d_2]}.$$

For a positive integer k, denote by $x^{\otimes k}$ the k-fold iterated tensor product of x with itself. We leave the proof of the following simple proposition as an exercise.

Proposition 2.2. *For any $x, y \in \mathbb{R}^d$, we have $\langle x^{\otimes k}, y^{\otimes k}\rangle = \langle x, y\rangle^k$.*

Proof of Theorem 1.1: Let

$$c = \sinh^{-1}(1) = \ln(1+\sqrt{2}).$$

Fix a positive integer d and two sets of d-dimensional unit vectors $x_1, \dots, x_n, y_1, \dots, y_n \in S_2^{d-1}$. We show that there exist $\{-1,1\}$-valued random variables $a_1, \dots, a_n$, $b_1, \dots, b_n$ such that

$$\mathbb{E}[a_i b_j] = \frac{2c}{\pi}\langle x_i, y_j\rangle$$

holds for all $i, j \in [n]$. To see why this suffices to prove the theorem, observe that by linearity of expectation,

$$\frac{2c}{\pi}\sum_{i,j=1}^{n} A_{ij}\langle x_i, y_j\rangle = \mathbb{E}\Big[\sum_{i,j=1}^{n} A_{ij} a_i b_j\Big] \le \max_{a_i, b_j \in \{-1,1\}} \sum_{i,j=1}^{n} A_{ij} a_i b_j.$$

This shows that $K_G \le \pi/(2c)$.

To obtain the random signs, define two new sequences of vectors:

$$u_i = \bigoplus_{k=1}^{\infty} \sqrt{|\alpha_k|} c^{k/2}\, x_i^{\otimes k}$$
$$v_j = \bigoplus_{k=1}^{\infty} \operatorname{sign}(\alpha_k)\, \sqrt{|\alpha_k|} c^{k/2}\, y_j^{\otimes k}.$$

Using the Taylor expansions of sin and sinh and Proposition 2.2 it is easy to verify that these are unit vectors and that

$$\langle u_i, v_j\rangle = \sin(c\langle x_i, v_j\rangle).$$

Observe that these are infinite-dimensional. However, since there are only $2n$ of them, they span a space of dimension at most $2n$, and it follows that there exist unit vectors $u_1', \dots, u_n'$, $v_1', \dots, v_n' \in S_2^{2n-1}$ such that $\langle u_i', v_j'\rangle = \langle u_i, v_j\rangle$ for all $i, j \in [n]$. Let $g = (g_1, \dots, g_{2n})$ be a random vector of independent standard normal random variables. Define

$$a_i = \operatorname{sign}(\langle u_i', g\rangle) \quad \text{and} \quad b_j = \operatorname{sign}(\langle v_j', g\rangle).$$

Then, by Lemma 2.1,

$$\begin{aligned}\frac{\pi}{2}\mathbb{E}[a_i b_j] &= \arcsin(\langle u_i', v_j'\rangle)\\ &= \arcsin(\sin(c\langle x_i, y_j\rangle))\\ &= c\langle x_i, y_j\rangle.\end{aligned}$$

This proves the theorem. □

3. Grothendieck factorization

In this section and the next, given a matrix $A \in \mathbb{R}^{m\times n}$ and normed vector spaces $X = (\mathbb{R}^n, \|\ \|_X)$, $Y = (\mathbb{R}^m, \|\ \|_Y)$, we shall write $\|A\|_{X\to Y}$ for the operator norm of the linear operator $X \to Y$ given by $x \mapsto Ax$.

While it is not hard to interpret the $\ell_\infty \to \ell_1$ norm, one may wonder what it means for a matrix A to satisfy $\|A\|_G \le 1$. The answer turns out to be that A can be "factored through a Hilbert space", meaning that there exist matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ such that A can decomposed as $A = BC$ and such that $\|B\|_{\ell_2\to\ell_1}\|C\|_{\ell_\infty\to\ell_2} \le 1$. This is a result of the following lemma, which we prove in this section.

Lemma 3.1 (Grothendieck)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a matrix without zero rows or columns. Then, satisfies $\|A\|_G \le C$ if and only if there exist positive unit vectors $u, v \in \mathbb{R}_{>0} \cap S_2^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,*

$$|\langle Ax, y\rangle| \le C\|x \circ u\|_2 \|y \circ v\|_2, \tag{3}$$

where $\circ$ denotes the entry-wise product.

Proof: We prove the "only if" direction and leave the "if" direction as an exercise. Let $M = A/\|A\|_G$, so that $\|M\|_G \le 1$. Then, by the AMGM inequality, it holds that for arbitrary vectors x_i, y_j of the same dimension, we have

$$\Big|\sum_{i,j=1}^n M_{ij}\langle x_i, y_j\rangle\Big| \le \max_{i,j\in[n]} \|x_i\|\|y_j\| \le \frac{1}{2}\max_{i,j\in[n]}(\|x_i\|^2 + \|y_j\|^2). \tag{4}$$

Define the set $K \subseteq \mathbb{R}^{n\times n}$ by

$$K = \left\{ \Big(\|x_i\|^2 + \|y_j\|^2 - 2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle x_k, y_\ell\rangle\Big|\Big)_{i,j=1}^n \,:\, d \in \mathbb{N},\ \ x_i, y_j \in \mathbb{R}^d \right\}.$$

We show that K is a convex cone. For every $t \in \mathbb{R}_+$ and matrix $Q \in K$ given by vectors x_i, y_j, the vectors $x_i' = \sqrt{t}x_i$ and $y_j' = \sqrt{t}y_j$ similarly define tQ, and so K is a cone. We now show that K is convex. Let

$Q, Q' \in K$ be specified by x_i, y_j and x'_i, y'_j respectively. Then, for any $\lambda \in [0,1]$, the vectors $(\sqrt{\lambda}x_i, \sqrt{1-\lambda}x'_i), (\sqrt{\lambda}y_j, \sqrt{1-\lambda}y'_j)$ can easily be seen to define $\lambda Q + (1-\lambda)Q'$, which thus also belongs to K.

Additionally, it follows from (4) that K is disjoint from the open convex cone $\mathbb{R}^{n\times n}_{<0}$ of matrices with strictly negative entries. It thus follows from the Hahn–Banach separation theorem that there is a nonzero matrix $L \in \mathbb{R}^{n\times n}$ such that $\langle L, Q\rangle \geq 0$ for all $Q \in K$ and $\langle L, N\rangle < 0$ for all $N \in \mathbb{R}^{n\times n}_{<0}$. The second inequality implies that L is entry-wise non-negative. Indeed, suppose that $L_{ij} \leq -\varepsilon$ for some $\varepsilon > 0$ and let $\delta = \max_{k,l} |L_{kl}|$. Consider the negative matrix N given by $N_{ij} = -1$ and $N_{kl} = -\varepsilon/(\delta n^2)$ for all $(k,l) \neq (i,j)$. Then,

$$\langle L, N\rangle \geq \varepsilon - \varepsilon(n^2-1)/n^2 > 0,$$

a contradiction.

Let $P = L/\sum_{ij} L_{ij}$, so that P defines a probability distribution on $[n]^2$. Then, for any $Q \in K$,

$$\begin{aligned} 0 \leq \langle P, Q\rangle &= \sum_{i,j=1}^n P_{ij}(\|x_i\|^2 + \|y_j\|^2) - 2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle x_k, y_\ell\rangle\Big| \\ &= \sum_{i=1}^n \sigma_i\|x_i\|^2 + \sum_{j=1}^n \mu_j\|y_j\|^2 - 2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle x_k, y_\ell\rangle\Big|, \end{aligned}$$

where $\sigma_i = P_{i1} + \cdots + P_{in}$ and $\mu_j = P_{1j} + \cdots + P_{nj}$. Rearranging the inequality above, it follows that for every $\lambda > 0$, we have

$$\begin{aligned} 2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle x_k, y_\ell\rangle\Big| &= 2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle \lambda x_k, \lambda^{-1} y_\ell\rangle\Big| \\ &\leq \lambda^2 \sum_{i=1}^n \sigma_i\|x_i\|_2^2 + \lambda^{-2}\sum_{j=1}^n \mu_j\|y_j\|_2^2. \end{aligned} \tag{5}$$

Setting

$$\lambda = \left(\frac{\sum_{j=1}^n \mu_j\|y_j\|_2^2}{\sum_{i=1}^n \sigma_i\|x_i\|_2^2}\right)^{1/4}$$

in (5), we find that

$$2\Big|\sum_{k,\ell=1}^n M_{k\ell}\langle x_k, y_\ell\rangle\Big| \leq 2\Big(\sum_{i=1}^n \sigma_i\|x_i\|_2^2\Big)^{1/2}\Big(\sum_{j=1}^n \mu_j\|y_j\|_2^2\Big)^{1/2}.$$

In particular, by letting $u_i = \sqrt{\sigma_i}$, $v_i = \sqrt{\mu_i}$ for every $i \in [n]$, the above shows that if $x_k, y_\ell \in \mathbb{R}$, then

$$|\langle Mx, y\rangle| \leq \|x \circ u\|_2 \|y \circ v\|_2. \tag{6}$$

The assumption that M has no zero rows or columns shows that u and v have strictly positive entries. To see this, first observe that u and v have at least one non-zero entry because they have unit ℓ_2-norm. Let $i \in [n]$ be arbitrary, let $j \in [n]$ be such that $M_{ij} \neq 0$ (as per the assumption on M) and let $k \in [n]$ such that $v_k \neq 0$. Setting $x = e_i$ and $y = \operatorname{sign}(M_{ij})e_j + \operatorname{sign}(M_{ik})e_k$ in (6) shows that

$$0 < |M_{ij}| \leq |M_{ij}| + |M_{ik}| = |\langle Mx, y\rangle| \leq u_i\sqrt{v_j^2 + v_k^2} \leq u_i v_j.$$

In particular, $u_i > 0$ for each $i \in [n]$. The same argument shows that v has strictly positive entries as well. □

Combining Lemma 3.1 with Theorem 1.1 (Grothendieck's inequality) gives the following factorization result. For $x \in \mathbb{R}^n$, denote by $\operatorname{Diag}(x)$ the matrix whose diagonal is x and whose off-diagonals are all zero.

Corollary 3.2. *For any matrix $A \in \mathbb{R}^{n\times n}$, there exist $B, C \in \mathbb{R}^{n\times n}$ such that $A = BC$ and $\|B\|_{\ell_2\to\ell_1}\|C\|_{\ell_\infty\to\ell_2} \leq K_G\|A\|_{\ell_\infty\to\ell_1}$.*

Proof: Let $u, v \in \mathbb{R}^n$ be as in Lemma 3.1. Since u, v are Euclidean unit vectors, it follows that $\|\operatorname{Diag}(u)\|_{\ell_\infty\to\ell_2} = \|\operatorname{Diag}(v)\|_{\ell_2\to\ell_1} \leq 1$. Define $M = \operatorname{Diag}(v)^{-1}A\operatorname{Diag}(u)^{-1}$. Then, it follows from Lemma 3.1 and Theorem 1.1 that for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}|\langle Mx, y\rangle| &\leq \|A\|_G\big\|\big(\operatorname{Diag}(u)^{-1}x\big)\circ u\big\|_2\big\|\big(\operatorname{Diag}(v)^{-1}y\big)\circ v\big\|_2\\ &\leq K_G\|A\|_{\ell_\infty\to\ell_1}\|x\|_2\|y\|_2.\end{aligned}$$

In particular, this shows that $\|M\|_{\ell_2\to\ell_2} \leq K_G\|A\|_{\ell_\infty\to\ell_1}$. Setting $B = \operatorname{Diag}(v)M$ and $C = \operatorname{Diag}(u)$ then gives the result since

$$\|B\|_{\ell_2\to\ell_1} \leq \|\operatorname{Diag}(v)\|_{\ell_2\to\ell_1}\|M\|_{\ell_2\to\ell_2}.$$

□

4. An application to geometric functional analysis

We end with a first application of Grothendieck's inequality. A finite-dimensional normed space X *embeds* into normed space Y with *distortion* C if there is a subspace $Z \subseteq Y$ with the same dimension as X such that $d(X, Z) \leq C$, where d is the Banach–Mazur distance.

Corollary 4.1. *Let $n \le k$ be positive integers and let $X = (\mathbb{R}^n, \| \ \|_X)$ be a normed space. Suppose that X and its dual X^* embed into ℓ_1^k with distortion C_1, C_2, respectively. Then, $d(X, \ell_2^n) \le K_G C_1 C_2$.*

A famous result of Dvoretzky, which we will see later in this course, implies that a converse of Corollary 4.1 also holds. In particular, there is a constant C such that if $d(X, \ell_2^n) \le C'$, then both X and X^* embed into ℓ_1^{2n} with distortion at most CC'. A more basic result due to Khitchine, which we will also see later, shows that this holds for $\ell_1^{2^n}$.

Proof: Let $S_1, S_2 \subseteq \ell_1^k$ be subspaces such that $d(X, S_1) \le C_1$ and $d(X^*, S_2) \le C_2$. Then, there exist $A_1, B_1, A_2, B_2 \in \mathbb{R}^{k\times n}$ such that $A_1^{\mathsf{T}} B_1 = A_2^{\mathsf{T}} B_2 = I_n$ and such that

$$\|A_1^{\mathsf{T}}\|_{S_1\to X}\|B_1\|_{X\to S_1} \le C_1 \quad \text{and} \quad \|A_2^{\mathsf{T}}\|_{S_2\to X^*}\|B_2\|_{X^*\to S_2} \le C_2.$$

We can thus write the identity (which should be thought of as a linear map from X to itself) as

$$(A_1^{\mathsf{T}} B_1)(A_2^{\mathsf{T}} B_2)^{\mathsf{T}} = A_1^{\mathsf{T}} B_1 B_2^{\mathsf{T}} A_2 = I_n. \tag{7}$$

Note that

$$\|B_1\|_{X\to\ell_1} = \|B_1\|_{X\to S_1}$$
$$\|B_2^{\mathsf{T}}\|_{\ell_\infty\to X} = \|B_2\|_{X^*\to\ell_1} = \|B_2\|_{X^*\to S_1}.$$

Since operator norms are sub-multiplicative, the matrix appearing in the middle of (7), $M = B_1 B_2^{\mathsf{T}} \in \mathbb{R}^{k\times k}$, therefore satisfies

$$\|M\|_{\ell_\infty\to\ell_1} \le \|B_1\|_{X\to\ell_1}\|B_2^{\mathsf{T}}\|_{\ell_\infty\to X} = \|B_1\|_{X\to S_1}\|B_2\|_{X^*\to S_2}.$$

It follows from Corollary 3.2 that there exist $M_1, M_2 \in \mathbb{R}^{k\times k}$ with which we can factor M as $M = M_1^{\mathsf{T}} M_2$ and which satisfy the norm bounds

$$\|M_1^{\mathsf{T}}\|_{\ell_2\to\ell_1}\|M_2\|_{\ell_\infty\to\ell_2} \le K_G\|M\|_{\ell_\infty\to\ell_1}.$$

Using this factorization of M, we can rewrite the factorization (7) as

$$I_n = A_1^{\mathsf{T}} M_1^{\mathsf{T}} M_2 A_2. \tag{8}$$

The image of $M_2 A_2$ is an n-dimensional subspace of $\mathbb{R}^k$. Therefore, there is an isometry from ℓ_2^n to ℓ_2^k given by a matrix $D \in \mathbb{R}^{k\times n}$ such that the right-hand side of (7) equals $(A_1^{\mathsf{T}} M_1^{\mathsf{T}} D)(D^{\mathsf{T}} M_2 A_2)$. This gives a factorization of the identity operator on X. In particular, since S_2^* is a quotient of ℓ_∞^n, we get that

$$\begin{aligned} d(X,\ell_2^n) &\le \|A_1^\mathsf{T} M_1^\mathsf{T} D\|_{\ell_2\to X}\|D^\mathsf{T} M_2 A_2\|_{X\to\ell_2} \\ &\le \|A_1^\mathsf{T}\|_{S_1\to X}\|M_1^\mathsf{T}\|_{\ell_2\to S_1}\|M_2\|_{S_2^*\to\ell_2}\|A_2\|_{X\to S_2^*} \\ &\le \|A_1^\mathsf{T}\|_{S_1\to X}\|M_1^\mathsf{T}\|_{\ell_2\to \ell_1}\|M_2\|_{\ell_\infty\to\ell_2}\|A_2\|_{X\to S_2^*} \\ &\le K_G\|A_1^\mathsf{T}\|_{S_1\to X}\|M\|_{\ell_\infty\to\ell_1}\|A_2^\mathsf{T}\|_{X^*\to S_2} \\ &\le K_G\|A_1^\mathsf{T}\|_{S_1\to X}\|B_1\|_{X\to S_1}\|A_2^\mathsf{T}\|_{S_2\to X^*}\|B_2\|_{X^*\to S_2} \\ &\le K_G C_1 C_2. \end{aligned}$$

□

5. Exercises

Exercise 5.1. Show that the quantities $\|A\|_{\ell_\infty\to\ell_1}$ and $\|A\|_G$ are norms.

Exercise 5.2. Let $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Show that $\|H\|_{\ell_\infty\to\ell_1} = 2$.

Exercise 5.3. Prove Proposition 2.2.

Exercise 5.4. Prove the "if" direction in Lemma 3.1.

References

[BMMN13] Mark Braverman, Konstantin Makarychev, Yury Makarychev, and Assaf Naor. The Grothendieck constant is strictly smaller than Krivine's bound. *Forum Math. Pi*, 1:e4, 42, 2013. Preliminary version in FOCS'11.

[Dav84] Alexander Davie. Lower bound for K_G. Unpublished, 1984.

[Gro53] Alexander Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques (French). *Bol. Soc. Mat. São Paulo*, 8:1–79, 1953.

[Kri79] J. L. Krivine. Constantes de Grothendieck et fonctions de type positif sur les sphères. *Adv. Math.*, 31:16–30, 1979.

[LP68] J. Lindenstrauss and A. Pełczyński. Absolutely summing operators in L_p-spaces and their applications. *Studia Math.*, 29:275–326, 1968.

[Pis12] Gilles Pisier. Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc.*, 49(2):237–323, 2012. also available at arXiv:1101.4195.

[Ree91] James Reeds. A new lower bound on the real Grothendieck constant. Manuscript (`http://www.dtc.umn.edu/~reedsj/bound2.dvi`), 1991.