GEOMETRIC FUNCTIONAL ANALYSIS AND APPLICATIONS

—LECTURE NOTES—
QUASIRANDOM GRAPHS

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1. Quasirandom graphs

In this lecture we will study an application of Grothendieck's inequality to quasirandom graph properties. A quasirandom graph property is a property one finds with high probability in a random graph where each edge is present with some fixed probability, independently of all others.

A graph is *d-regular* if each vertex is contained in exactly d edges. For a graph $G = (V, E)$ and subsets $S, T \subseteq V$, denote by $e(S, T)$ the number of edges with an endpoint in both S and T. The adjacency matrix of G is the matrix $A \in \mathbb{R}^{V \times V}$ given by $A_{u,v} = e(\{u\}, \{v\})$. The particular quasirandom parameters we will focus on are as follows.

Definition 1.1. An n-vertex d-regular graph $G = (V, E)$ is *ε-uniform* if for all vertex-subsets $S, T \subseteq V$, we have

$$\left| \frac{e(S,T)}{dn} - \frac{|S||T|}{n^2} \right| \leq \varepsilon. \tag{1}$$

Denote by $\Delta(G)$ the smallest ε such that G is ε-uniform.

Definition 1.2. A graph G is an (n, d, λ)-graph if it has n vertices, degree d and all the eigenvalues of its adjacency matrix, except the largest, are at most λ in absolute value. Denote by $\lambda(G)$ the smallest λ such that G is an (n, d, λ)-graph.

It turns out that for any constant $c > 0$ and $d = cn$, then as $n \to \infty$, a random d-regular graph will satisfy both $\Delta(G) = o(1)$ and $\lambda(G) = o(d)$ with high probability. For deterministic graphs, we have the following basic result.

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Lemma 1.3 (Expander mixing lemma)**.** *Let $G = (V, E)$ be an (n, d, λ)-graph. Then, for all vertex-subsets $S, T \subseteq V$, we have*

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|}. \tag{2}$$

Since the right-hand side of (2) is at most λn, dividing both sides by dn shows that an (n, d, λ)-graph is (λd^{-1})-uniform. A famous result of Chung, Graham and Wilson [CGW89] shows that the converse of Lemma 1.3 holds for dense graphs (in which case $d \geq \Omega(n)$). In particular, for any $\delta > 0$, there is a $C(\delta) > 0$ such any n-vertex d-regular ε-uniform graph with $d \geq \delta n$ is an (n, d, λ)-graph with $\lambda \leq C(\delta)\varepsilon d$.

For sparse graphs, such a converse no longer holds in general. It was shown recently by Conlon and Zhao in [CZ17] that there exist d-regular n-vertex graphs with $d \to \infty$ as $n \to \infty$ that are $o(1)$-uniform, but for which $\lambda \geq \Omega(d)$. However, in the same paper they also shows that this situation cannot occur for sparse graphs with a sufficient amount of symmetry. For a graph $G = (V, E)$, an *automorphism* is a permutation $\pi : V \to V$ such that $\{\pi(u), \pi(v)\} \in E$ if and only if $\{u, v\} \in E$.

Definition 1.4. A graph G is *vertex transitive* if for every pair of vertices u, v, there exists an automorphism π such that $\pi(u) = v$.

Theorem 1.5 (Conlon–Zhao)**.** *Any n-vertex d-regular graph that is vertex transitive and ε-uniform is an (n, d, λ)-graph for $\lambda \leq 4\varepsilon K_G d$, where $K_G \in [1, \infty)$ is the Grothendieck constant.*

Bilu and Linial [BL06] proved that if a d-regular graph satisfies the stronger condition that the left-hand side of (1) is at most $\varepsilon d\sqrt{|S||T|}$, then $\lambda \leq C\varepsilon d \log(2/\varepsilon)$ for some absolute constant $C > 0$. Theorem 1.5 gives the stronger conclusion $\lambda \leq 4\varepsilon K_G d$ from the weaker condition (1).

2. A link with Grothendieck's inequality

The proof of Theorem 1.5 uses Grothendieck's inequality.

Theorem 2.1 (Grothendieck's inequality)**.** *There exists an absolute constant $K_G \in (0, \infty)$ such that the following holds. For any positive integer n and matrix $B \in \mathbb{R}^{n\times n}$, we have*

$$\|B\|_G \leq K_G \|B\|_{\infty\to 1}. \tag{3}$$

For an n-vertex d-regular graph G with adjacency matrix A, we use two simple propositions. Let J denotes the all-ones matrix.

Proposition 2.2. *Let G be an n-vertex d-regular graph and let A be its adjacency matrix. Let $B = A - \frac{d}{n}J$. Then,*

$$\lambda(G) = \|B\| \tag{4}$$

$$\|B\|_{\infty\to 1} \leq 4dn\Delta(G). \tag{5}$$

The following key lemma allows us to apply Grothendieck's inequality.

Lemma 2.3. *Let G be a vertex-transitive n-vertex d-regular graph and let A be its adjacency matrix. Let $B = A - \frac{d}{n}J$. Then,*

$$n\|B\| \leq \|B\|_G. \tag{6}$$

Proof of Theorem 1.5: Let A be the adjacency matrix of G and let $B = A - \frac{d}{n}J$. Then, by Proposition 2.2 and Lemma 2.3,

$$n\lambda(G) \overset{(4)}{=} n\|B\| \overset{(6)}{\leq} \|B\|_G \overset{(3)}{\leq} K_G\|B\|_{\infty\to 1} \overset{(5)}{\leq} 4dnK_G\Delta(G).$$

□

3. Proof of Lemma 2.3

The following proof of Lemma 2.3, which is even shorter than the original, uses Grothendieck's factorization lemma.

Lemma 3.1 (Grothendieck)**.** *For any matrix $A \in \mathbb{R}^{n\times n}$ without zero rows or columns, there exist positive unit vectors $u, v \in \mathbb{R}_{>0} \cap S^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,*

$$|\langle Ax, y\rangle| \leq \|A\|_G \, \|x \circ u\|_2 \|y \circ v\|_2. \tag{7}$$

For a permutation $\pi \in S_n$ and $A \in \mathbb{R}^{n\times n}$, let $A^\pi = (A_{\pi(i),\pi(j)})_{i,j=1}^n$. Observe that if A is the adjacency matrix of a graph G and π is an automorphism of G, then $A^\pi = A$.

Proof of Lemma 2.3: Define $C = B/\|B\|_G$, so that $\|C\|_G = 1$. Since C is symmetric, we have $\|C\| = \max\{|\langle Cx, x\rangle| : x \in S_2^{n-1}\}$. Moreover, since G is regular, C has no zero rows or columns. Therefore, by Lemma 3.1 and the AMGM inequality, there exist positive unit vectors $u, v \in \mathbb{R}_{>0} \cap S_2^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,

$$|\langle Cx, y\rangle| \leq \|x \circ u\|_2 \|y \circ v\|_2 \leq \frac{1}{2}\big(\|x \circ u\|_2^2 + \|y \circ v\|_2^2\big).$$

Fix $x, y \in \mathbb{R}^n$. Let $\Gamma \leq S_n$ be the group of automorphisms of G and let Γ act on $\mathbb{R}^n$ in the natural way. Since the adjacency matrix A

satisfies $A^\pi = A$ for every $\pi \in \Gamma$, it follows that $C^\pi = C$ for every $\pi \in \Gamma$, and therefore,

$$\langle Cx, y\rangle = \langle C^\pi x, y\rangle = \langle C(\pi^{-1}x), (\pi^{-1}y)\rangle.$$

Putting the above two observations together gives

$$\begin{aligned}|\langle Cx, y\rangle| &= \big|\mathbb{E}_{\pi\in\Gamma}\big[\langle C(\pi^{-1}x), (\pi^{-1}y)\rangle\big]\big| \\ &\le \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|(\pi^{-1}x)\circ u\|_2^2\big] + \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|(\pi^{-1}y)\circ v\|_2^2\big] \\ &= \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|x\circ(\pi u)\|_2^2\big] + \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|y\circ(\pi v)\|_2^2\big].\end{aligned}$$

Since Γ is transitive and u is a unit vector, the first expectation on the last line equals

$$\mathbb{E}_{\pi\in\Gamma}\Big[\sum_{i=1}^n x_i^2 u_{\pi(i)}^2\Big] = \sum_{i=1}^n x_i^2 \mathbb{E}_{\pi\in\Gamma}\big[u_{\pi(i)}^2\big] = n^{-1}\|x\|_2^2.$$

Applying the same argument to the second expectation gives

$$|\langle Cx, y\rangle| \le \frac{1}{2n}\big(\|x\|_2^2 + \|y\|_2^2\big).$$

It follows that $\|C\| \le 1/n$, which gives the claim. □

4. Optimality of Conlon–Zhao

There is a sense in which Theorem 1.5 is optimal. To see what this means, we pass to infinite weighted graphs whose vertex set consists of a high-dimensional Euclidean unit sphere. Let ω_n denote the Haar probability measure on S_2^{n-1}. For a continuous function $f \in C(S_2^{n-1})$, define its L_2-norm by

$$\|f\|_{L_2} = \Big(\int_{S_2^{n-1}} |f(x)|^2 d\omega_n(x)\Big)^{\frac{1}{2}}.$$

For a bilinear form $B : C(S_2^{n-1}) \times C(S_2^{n-1}) \to \mathbb{R}$, define

$$\|B\|_{L_2\to L_2} = \sup\{|B(f,g)| : \|f\|_{L_2} \le 1,\ \|g\|_{L_2} \le 1\}.$$

Also define its operator norm in the natural way, by

$$\|B\| = \sup\{|B(f,g)| : \|f\|_{L_\infty} \le 1,\ \|g\|_{L_\infty} \le 1\},$$

where $\|f\|_{L_\infty} = \sup_{x\in S_2^{n-1}} |f(x)|$ is the supremum norm. Observe that $\|B\| \le \|B\|_{L_2\to L_2}$. In the earlier section, we had the dimension factor of n in the analogous inequality. The reason why this factor does not

appear here is that the L_2 norm is taken with respect to a probability measure as opposed to a counting measure.

Given a permutation $\pi : S_2^{n-1} \to S_2^{n-1}$ and a function $f \in C(S_2^{n-1})$, let f^π be the function given by $f^\pi(x) = f(\pi(x))$. Define the bilinear form $B^\pi : C(S_2^{n-1}) \times C(S_2^{n-1}) \to \mathbb{R}$ by $B^\pi(f,g) = B(f^\pi, g^\pi)$. The permutation π is an *automorphism* of B if $B^\pi = B$. Finally, say that B is *transitive* if for any two points $x, y \in S_2^{n-1}$, there is an automorphism π of B such that $\pi(x) = y$.

Lemma 4.1. *For every positive integer n and matrix $A \in \mathbb{R}^{n\times n}$, there is a transitive bilinear form $B : C(S_2^{2n-1}) \times C(S_2^{2n-1}) \to \mathbb{R}$ such that*

$$\frac{\|B\|_{L_2\to L_2}}{\|B\|} \geq \frac{\|A\|_G}{\|A\|_{\ell_\infty\to\ell_1}}.$$

Proof: Let μ_n denote the Haar probability measure on the orthogonal group $O(n)$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in S_2^{2n-1}$ be some Euclidean unit vectors. Define B by

$$B(f,g) = \sum_{i,j=1}^n A_{ij} \int_{O(n)} f(Ux_i)g(Uy_j)d\mu_n(U).$$

By invariance of the μ_n under $O(n)$ itself, it follows that for every $V \in O(n)$, we have $B^V = V$. In particular, B is transitive.

For any $f, g \in C(S^{2n-1}$ such that $\|f\|_{L_\infty} \leq 1$ and $\|g\|_{L_\infty} \leq 1$, we have

$$|B(f,g)| \leq \int_{O(n)} \Big| \sum_{i,j=1}^n A_{ij} f(Ux_i)g(Uy_j)\Big| d\mu_n(U) \leq \|A\|_{\ell_\infty\to\ell_1}.$$

In particular, $\|B\| \leq \|A\|_{\ell_\infty\to\ell_1}$.

For $k = 1, \ldots, 2n$, define $f_k \in C(S_2^{2n-1})$ by $f_k(x) = x_k$. Then,

$$\begin{aligned}\sum_{i=1}^{2n} B(f_i, f_i) &= \sum_{k=1}^{2n}\sum_{i,j=1}^n A_{ij} \int_{O(n)} (Ux)_k (Uy)_k d\mu_n(U) \\ &= \sum_{i,j=1}^n A_{ij} \int_{O(n)} \langle Ux, Uy\rangle \mu_n(U) \\ &= \sum_{i,j=1}^n A_{ij}\langle x_i, y_j\rangle.\end{aligned}$$

At the same time,

$$\begin{aligned}\Big|\sum_{k=1}^{2n} B(f_k, f_k)\Big| &\leq \|B\|_{L_2\to L_2} \sum_{k=1}^{n} \|f_k\|_{L_2}^2 \\ &= \|B\|_{L_2\to L_2} \int_{S_2^{2n-1}} \sum_{k=1}^{2n} x_k^2 d\omega_n(x) \\ &= \|B\|_{L_2\to L_2}.\end{aligned}$$

Since the x_i, y_j where arbitrary, it follows that $\|B\|_{L_2\to L_2} \geq \|A\|_G$. $\square$

5. Exercises

Exercise 5.1. Prove Lemma 1.3 (the Expander mixing lemma).

Exercise 5.2. Prove Proposition 2.2.

Exercise 5.3. Show that in fact equality holds in (5). [Hint: use the fact that both the rows and the columns of $A - \frac{d}{n}J$ sum to zero.]

Exercise 5.4. Show that equality holds in Lemma 2.3.

References

[BL06] Yonatan Bilu and Nathan Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.

[CGW89] Fan R. K. Chung, Ronald L. Graham, and Richard M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[CZ17] David Conlon and Yufei Zhao. Quasirandom cayley graphs. *Discrete Analysis*, 6, 2017.