1. Quasirandom graphs

In this lecture we will study an application of Grothendieck’s inequality to quasirandom graph properties. A quasirandom graph property is a property one finds with high probability in a random graph where each edge is present with some fixed probability, independently of all others.

A graph is $d$-regular if each vertex is contained in exactly $d$ edges. For a graph $G = (V, E)$ and subsets $S, T \subseteq V$, denote by $e(S, T)$ the number of edges with an endpoint in both $S$ and $T$. The adjacency matrix of $G$ is the matrix $A \in \mathbb{R}^{V \times V}$ given by $A_{u,v} = e\{\{u\}, \{v\}\}$. The particular quasirandom parameters we will focus on are as follows.

**Definition 1.1.** An $n$-vertex $d$-regular graph $G = (V, E)$ is $\varepsilon$-uniform if for all vertex-subsets $S, T \subseteq V$, we have

$$
\left| \frac{e(S, T)}{dn} - \frac{|S||T|}{n^2} \right| \leq \varepsilon.
$$

Denote by $\Delta(G)$ the smallest $\varepsilon$ such that $G$ is $\varepsilon$-uniform.

**Definition 1.2.** A graph $G$ is an $(n, d, \lambda)$-graph if it has $n$ vertices, degree $d$ and all the eigenvalues of its adjacency matrix, except the largest, are at most $\lambda$ in absolute value. Denote by $\lambda(G)$ the smallest $\lambda$ such that $G$ is an $(n, d, \lambda)$-graph.

It turns out that for any constant $c > 0$ and $d = cn$, then as $n \to \infty$, a random $d$-regular graph will satisfy both $\Delta(G) = o(1)$ and $\lambda(G) = o(d)$ with high probability. For deterministic graphs, we have the following basic result.
Lemma 1.3 (Expander mixing lemma). Let \( G = (V, E) \) be an \((n, d, \lambda)\)-graph. Then, for all vertex-subsets \( S, T \subseteq V \), we have

\[
\left| e(S, T) - \frac{d}{n}|S||T| \right| \leq \lambda \sqrt{|S||T|}.
\]

Since the right-hand side of (2) is at most \( \lambda n \), dividing both sides by \( dn \) shows that an \((n, d, \lambda)\)-graph is \((\lambda d - 1)\)-uniform. A famous result of Chung, Graham and Wilson [CGW89] shows that the converse of Lemma 1.3 holds for dense graphs (in which case \( d \geq \Omega(n) \)). In particular, for any \( \delta > 0 \), there is a \( C(\delta) > 0 \) such any \( n \)-vertex \( d \)-regular \( \varepsilon \)-uniform graph with \( d \geq \delta n \) is an \((n, d, \lambda)\)-graph with \( \lambda \leq C(\delta)\varepsilon d \).

For sparse graphs, such a converse no longer holds in general. It was shown recently by Conlon and Zhao in [CZ17] that there exist \( d \)-regular \( n \)-vertex graphs with \( d \to \infty \) as \( n \to \infty \) that are \( o(1) \)-uniform, but for which \( \lambda \geq \Omega(d) \). However, in the same paper they also shows that this situation cannot occur for sparse graphs with a sufficient amount of symmetry. For a graph \( G = (V, E) \), an automorphism is a permutation \( \pi : V \to V \) such that \( \{\pi(u), \pi(v)\} \in E \) if and only if \( \{u, v\} \in E \).

Definition 1.4. A graph \( G \) is vertex transitive if for every pair of vertices \( u, v \), there exists an automorphism \( \pi \) such that \( \pi(u) = v \).

Theorem 1.5 (Conlon–Zhao). Any \( n \)-vertex \( d \)-regular graph that is vertex transitive and \( \varepsilon \)-uniform is an \((n, d, \lambda)\)-graph for \( \lambda \leq 4\varepsilon K_G d \), where \( K_G \in [1, \infty) \) is the Grothendieck constant.

Bilu and Linial [BL06] proved that if a \( d \)-regular graph satisfies the stronger condition that the left-hand side of (1) is at most \( \varepsilon d \sqrt{|S||T|} \), then \( \lambda \leq C\varepsilon d \log(2/\varepsilon) \) for some absolute constant \( C > 0 \). Theorem 1.5 gives the stronger conclusion \( \lambda \leq 4\varepsilon K_G d \) from the weaker condition (1).

2. A link with Grothendieck’s inequality

The proof of Theorem 1.5 uses Grothendieck’s inequality.

Theorem 2.1 (Grothendieck’s inequality). There exists an absolute constant \( K_G \in (0, \infty) \) such that the following holds. For any positive integer \( n \) and matrix \( B \in \mathbb{R}^{n \times n} \), we have

\[
\|B\|_G \leq K_G \|B\|_{\infty \to 1}.
\]

For an \( n \)-vertex \( d \)-regular graph \( G \) with adjacency matrix \( A \), we use two simple propositions. Let \( J \) denotes the all-ones matrix.
Proposition 2.2. Let \( G \) be an \( n \)-vertex \( d \)-regular graph and let \( A \) be its adjacency matrix. Let \( B = A - \frac{d}{n} J \). Then,

\[
\lambda(G) = \|B\| \quad (4)
\]

\[
\|B\|_{\infty \rightarrow 1} \leq 4dn\Delta(G) \quad (5)
\]

The following key lemma allows us to apply Grothendieck’s inequality.

Lemma 2.3. Let \( G \) be a vertex-transitive \( n \)-vertex \( d \)-regular graph and let \( A \) be its adjacency matrix. Let \( B = A - \frac{d}{n} J \). Then,

\[
n\|B\| \leq \|B\|_G \quad (6)
\]

Proof of Theorem 1.5: Let \( A \) be the adjacency matrix of \( G \) and let \( B = A - \frac{d}{n} J \). Then, by Proposition 2.2 and Lemma 2.3,

\[
n\lambda(G) \overset{(4)}{=} n\|B\| \overset{(6)}{\leq} \|B\|_G \overset{(3)}{\leq} K_G\|B\|_{\infty \rightarrow 1} \overset{(5)}{\leq} 4dnK_G\Delta(G).
\]

\[
\square
\]

3. Proof of Lemma 2.3

The following proof of Lemma 2.3, which is even shorter than the original, uses Grothendieck’s factorization lemma.

Lemma 3.1 (Grothendieck). For any matrix \( A \in \mathbb{R}^{n \times n} \) without zero rows or columns, there exist positive unit vectors \( u, v \in \mathbb{R}^n \cap S^{n-1} \) such that for any \( x, y \in \mathbb{R}^n \),

\[
|\langle Ax, y \rangle| \leq \|A\|_G \|x \circ u\|_2 \|y \circ v\|_2.
\]

(7)

For a permutation \( \pi \in S_n \) and \( A \in \mathbb{R}^{n \times n} \), let \( A^\pi = (A_{\pi(i),\pi(j)})_{i,j=1}^n \). Observe that if \( A \) is the adjacency matrix of a graph \( G \) and \( \pi \) is an automorphism of \( G \), then \( A^\pi = A \).

Proof of Lemma 2.3: Define \( C = B/\|B\|_G \), so that \( \|C\|_G = 1 \). Since \( C \) is symmetric, we have \( \|C\| = \max\{|\langle Cx, x \rangle| : x \in S^{n-1}_2 \} \). Moreover, since \( G \) is regular, \( C \) has no zero rows or columns. Therefore, by Lemma 3.1 and the AMGM inequality, there exist positive unit vectors \( u, v \in \mathbb{R}^n \), such that for any \( x, y \in \mathbb{R}^n \),

\[
|\langle Cx, y \rangle| \leq \|x \circ u\|_2 \|y \circ v\|_2 \leq \frac{1}{2}(\|x \circ u\|_2^2 + \|y \circ v\|_2^2).
\]

Fix \( x, y \in \mathbb{R}^n \). Let \( \Gamma \leq S_n \) be the group of automorphisms of \( G \) and let \( \Gamma \) act on \( \mathbb{R}^n \) in the natural way. Since the adjacency matrix \( A \)
satisfies $A^\pi = A$ for every $\pi \in \Gamma$, it follows that $C^\pi = C$ for every $\pi \in \Gamma$, and therefore,

$$\langle Cx, y \rangle = \langle C^\pi x, y \rangle = \langle C(\pi^{-1}x), (\pi^{-1}y) \rangle.$$ 

Putting the above two observations together gives

$$|\langle Cx, y \rangle| = \left| \mathbb{E}_{\pi \in \Gamma} \left[ \langle C(\pi^{-1}x), (\pi^{-1}y) \rangle \right] \right| \leq \frac{1}{2} \mathbb{E}_{\pi \in \Gamma} [\| (\pi^{-1}x) \circ u \|^2_2] + \frac{1}{2} \mathbb{E}_{\pi \in \Gamma} [\| (\pi^{-1}y) \circ v \|^2_2] = \frac{1}{2} \mathbb{E}_{\pi \in \Gamma} [\| x \circ (\pi u) \|^2_2] + \frac{1}{2} \mathbb{E}_{\pi \in \Gamma} [\| y \circ (\pi v) \|^2_2].$$

Since $\Gamma$ is transitive and $u$ is a unit vector, the first expectation on the last line equals

$$\mathbb{E}_{\pi \in \Gamma} \left[ \sum_{i=1}^{n} x_i^2 u_{\pi(i)}^2 \right] = \sum_{i=1}^{n} x_i^2 \mathbb{E}_{\pi \in \Gamma} [u_{\pi(i)}^2] = n^{-1} \| x \|^2_2.$$

Applying the same argument to the second expectation gives

$$|\langle Cx, y \rangle| \leq \frac{1}{2n} (\| x \|^2_2 + \| y \|^2_2).$$

It follows that $\| C \| \leq 1/n$, which gives the claim. \qed

4. Optimality of Conlon–Zhao

There is a sense in which Theorem 1.5 is optimal. To see what this means, we pass to infinite weighted graphs whose vertex set consists of a high-dimensional Euclidean unit sphere. Let $\omega_n$ denote the Haar probability measure on $S^{n-1}_2$. For a continuous function $f \in C(S^{n-1}_2)$, define its $L^2$-norm by

$$\| f \|_{L^2} = \left( \int_{S^{n-1}_2} |f(x)|^2 d\omega_n(x) \right)^{\frac{1}{2}}.$$

For a bilinear form $B : C(S^{n-1}_2) \times C(S^{n-1}_2) \rightarrow \mathbb{R}$, define

$$\| B \|_{L^2 \rightarrow L^2} = \sup \{ |B(f, g)| : \| f \|_{L^2} \leq 1, \| g \|_{L^2} \leq 1 \}.$$

Also define its operator norm in the natural way, by

$$\| B \| = \sup \{ |B(f, g)| : \| f \|_{L^\infty} \leq 1, \| g \|_{L^\infty} \leq 1 \},$$

where $\| f \|_{L^\infty} = \sup_{x \in S^{n-1}_2} |f(x)|$ is the supremum norm. Observe that $\| B \| \leq \| B \|_{L^2 \rightarrow L^2}$.

In the earlier section, we had the dimension factor of $n$ in the analogous inequality. The reason why this factor does not
appear here is that the $L_2$ norm is taken with respect to a probability measure as opposed to a counting measure.

Given a permutation $\pi : S_2^{n-1} \to S_2^{n-1}$ and a function $f \in C(S_2^{n-1})$, let $f^\pi$ be the function given by $f^\pi(x) = f(\pi(x))$. Define the bilinear form $B^\pi : C(S_2^{n-1}) \times C(S_2^{n-1}) \to \mathbb{R}$ by $B^\pi(f, g) = B(f^\pi, g^\pi)$. The permutation $\pi$ is an automorphism of $B$ if $B^\pi = B$. Finally, say that $B$ is transitive if for any two points $x, y \in S_2^{n-1}$, there is an automorphism $\pi$ of $B$ such that $\pi(x) = y$.

**Lemma 4.1.** For every positive integer $n$ and matrix $A \in \mathbb{R}^{n \times n}$, there is a transitive bilinear form $B : C(S_2^{2n-1}) \times C(S_2^{2n-1}) \to \mathbb{R}$ such that

$$\frac{\|B\|_{L_2 \to L_2}}{\|B\|} \geq \frac{\|A\|_G}{\|A\|_{\ell_\infty \to \ell_1}}.$$

**Proof:** Let $\mu_n$ denote the Haar probability measure on the orthogonal group $O(n)$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in S_2^{2n-1}$ be some Euclidean unit vectors. Define $B$ by

$$B(f, g) = \sum_{i,j=1}^n A_{ij} \int_{O(n)} f(Ux_i)g(Uy_j) d\mu_n(U).$$

By invariance of the $\mu_n$ under $O(n)$ itself, it follows that for every $V \in O(n)$, we have $B^V = V$. In particular, $B$ is transitive.

For any $f, g \in C(S_2^{2n-1})$ such that $\|f\|_{L_\infty} \leq 1$ and $\|g\|_{L_\infty} \leq 1$, we have

$$|B(f, g)| \leq \int_{O(n)} \left| \sum_{i,j=1}^n A_{ij} f(Ux_i)g(Uy_j) \right| d\mu_n(U) \leq \|A\|_{\ell_\infty \to \ell_1}.$$

In particular, $\|B\| \leq \|A\|_{\ell_\infty \to \ell_1}$.

For $k = 1, \ldots, 2n$, define $f_k \in C(S_2^{2n-1})$ by $f_k(x) = x_k$. Then,

$$\sum_{i=1}^{2n} B(f_i, f_i) = \sum_{k=1}^{2n} \sum_{i,j=1}^n A_{ij} \int_{O(n)} (Ux)_k(Uy)_k d\mu_n(U)$$

$$= \sum_{i,j=1}^n A_{ij} \int_{O(n)} \langle Ux, Uy \rangle \mu_n(U)$$

$$= \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle.$$
At the same time,
\[ \left| \sum_{k=1}^{2n} B(f_k, f_k) \right| \leq \| B \|_{L_2 \to L_2} \sum_{k=1}^{n} \| f_k \|_{L_2}^2 \]
\[ = \| B \|_{L_2 \to L_2} \int_{S^{2n-1}} \sum_{k=1}^{2n} x_k^2 d\omega_n(x) \]
\[ = \| B \|_{L_2 \to L_2}. \]

Since the \( x_i, y_j \) where arbitrary, it follows that \( \| B \|_{L_2 \to L_2} \geq \| A \|_G. \) \( \square \)

5. EXERCISES

**Exercise 5.1.** Prove Lemma 1.3 (the Expander mixing lemma).

**Exercise 5.2.** Prove Proposition 2.2.

**Exercise 5.3.** Show that in fact equality holds in (5). [Hint: use the fact that both the rows and the columns of \( A - \frac{d}{n} J \) sum to zero.]

**Exercise 5.4.** Show that equality holds in Lemma 2.3.

REFERENCES

