Definitions:
For a non-singular matrix $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_k) \in \mathbb{R}^{n \times k}$ define the parallelepiped

$$\mathcal{P}(\mathbf{b}_1, \ldots, \mathbf{b}_k) := \mathcal{P}(B) = B[0,1)^k.$$

For a lattice $\mathcal{L} \subset \mathbb{R}^n$ define its (closed) Voronoi cell as

$$\mathcal{V}(\mathcal{L}) := \{\mathbf{x} \in \mathrm{span}(\mathcal{L}) : \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|, \forall \mathbf{y} \in \mathcal{L} \setminus \{\mathbf{0}\}\}.$$

Exercise 1 Let $\mathcal{L} \subset \mathbb{R}^n$ denote a full rank lattice. Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathcal{L}$ be linearly independent vectors. Show $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are a basis of $\mathcal{L}$ if and only if $\mathcal{P}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \cap \mathcal{L} = \{\mathbf{0}\}$.

Exercise 2 Let $\mathcal{L}_1 \subset \mathbb{R}^n$ be full rank lattice and let $\mathcal{L}_2 \subset \mathcal{L}_1$ denote a full rank sublattice. Show that the index $[\mathcal{L}_1 : \mathcal{L}_2] = \frac{\det(\mathcal{L}_2)}{\det(\mathcal{L}_1)}$ (recall at $[\mathcal{L}_1 : \mathcal{L}_2] = |\mathcal{L}_1 / \mathcal{L}_2|$).
(Hint: Take a fundamental domains D_1, D_2 of $\mathcal{L}_1, \mathcal{L}_2$ respectively, and write the volume of D_2 as an integral over lattice point counts over D_1.)

Exercise 3 Let $\mathcal{L} \subset \mathbb{R}^n$ be a full rank lattice with basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Let $U \sim \mathrm{uniform}(\mathcal{P}(\mathbf{b}_1, \ldots, \mathbf{b}_n))$ be uniformly distributed over the fundamental parallelepiped. Let $Z = U - \mathrm{CVP}(\mathcal{L}, U)$, where $\mathrm{CVP}(\mathcal{L}, U)$ denotes the closest vector to U in $\mathcal{L}$ (note that this is well-defined w.p. 1). Show that Z is uniformly distributed over the Voronoi cell $\mathcal{V}(\mathcal{L})$.

Exercise 4 Let $\mathcal{L} \subset \mathbb{R}^n$ denote a full rank lattice. Define

$$\lambda_1(B_1^n, \mathcal{L}) := \min_{\mathbf{y} \in \mathcal{L} \setminus \{\mathbf{0}\}} \|\mathbf{y}\|_1,$$

where $\|\mathbf{y}\|_1 = \sum_{i=1}^n |y_i|$ is the ℓ_1 norm and $B_1^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 \leq 1\}$ is the ℓ_1 ball.
Show that $\lambda_1(B_1^n, \mathcal{L}) \leq n \cdot \det(\mathcal{L})^{1/n}$.
(Hint: Minkowski's Convex Body Theorem)