Exercise 1 With respect to the ℓ_2-norm, prove that the covering radius $\mu(\Lambda)$ of Λ satisfyies $\mu(\Lambda)^2 \leq \sum_i \lambda_i(\Lambda)^2$.

Exercise 2 (Planetary Alignment) In a galaxy, far far away, there is a planetary system with $n+1$ planets, all orbiting a star in perfect circles. The planets have constant angular momentums $\sigma_0, \ldots, \sigma_n$ respectively. We say that the planets are in ε-planetary alignment when the angle of planet $1, \ldots, n$ with planet 0 is at most $2\pi\varepsilon$.

Suppose the planetary system is at 0-planetary alignment at time $t = 0$. Show that the next ε-planetary alignment occurs at time at most $2^n/(\varepsilon^{n-1} \max_{i \neq 0} |\sigma_i - \sigma_0|)$.

Exercise 3 (Optimal covering) Prove that the Hexagonal lattice is also optimal for covering (for the ℓ_2 norm), that is, over all lattices Λ of dimension 2 it minimizes $\mu(\Lambda)/\sqrt{\det(\Lambda)}$.

1. Express the covering radius in terms of the Voronoi cell.
2. Prove that the Voronoi cell of any lattice is a polygon with either 4 of 6 sides.
   (a) Take a basis $\mathbf{b}_1, \mathbf{b}_2$ as in the Wristwatch lemma: $\mathbf{b}_1 = (1, 0)$, $\mathbf{b}_2 = (\alpha, \beta)$, where $|\alpha| \leq 1/2$ and $|\beta| \geq \sqrt{3/4}$. Why can you assume without loss of generality that $\alpha \geq 0$ and $\beta \geq 0$?
   (b) Quickly treat the case $\alpha = 0$, to assume $0 < \alpha \leq 1/2$.
   (c) Shows that the frontier of the voronoi cell contains a non-trivial segment from $(1/2, \delta)$ to $(1/2, -\delta)$ for some $\delta > 0$.
   (d) Where does this segment ends ? (A qualitative answer suffices)
   (e) Draw a picture of your lattice, including all the shifted copies of that segment. Conclude exploiting the convexity of the Voronoi cell.
3. Relax the problem: show that among all symmetric hexagons included in the unit disc (not necessary built as a Voronoi cell), the regular hexagon inscribed in the unit circle has the maximal area.
   (a) If a symmetric hexagon is not inscribed in the unit circle, show that you can increase its area.
   (b) Express the area of a symmetric unit-inscribed hexagon $ABC(-A)(-B)(-C)$ as a function of the angles $\alpha_1 = \widehat{A0B}$, $\alpha_2 = \widehat{B0C}$, $\alpha_3 = \widehat{C0(-A)}$.
   (c) Note $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Conclude that the maximal area is reached when $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.