Exercise 1 (Properties of HKZ bases)
Let $\mathcal{L} \subset \mathbb{R}^n$ be a full rank lattice and let $\mathbf{B}$ be an HKZ-reduced basis of $\mathcal{L}$. That is, $\mathbf{B}$ is size-reduced and $\|\widetilde{\mathbf{b}}_i\| = \lambda_1(\pi_i(\mathcal{L}))$, where π_i is the projection onto $\text{span}(\mathbf{b}_1, \ldots, \mathbf{b}_{i-1})^\perp$.

Prove the following inequalities:

1. $\|\widetilde{\mathbf{b}}_i\| \le \lambda_i(\mathcal{L})$ for $i \in [n]$.
2. $\|\widetilde{\mathbf{b}}_i\| \le \|\mathbf{b}_j\|$ for $1 \le i \le j \le n$.
3. $\frac{1}{\sqrt{\frac{i-1}{4}+1}} \cdot \|\mathbf{b}_i\| \le \lambda_i(\mathcal{L}) \le \sqrt{\frac{i-1}{4}+1} \cdot \|\mathbf{b}_i\|, \forall i \in [n]$.
4. $\|\widetilde{\mathbf{b}}_i\| \le 2^{O(\log^2(j-i))} \|\widetilde{\mathbf{b}}_j\|, \forall 1 \le i < j \le n$.
   (Hint: Use the fact $\mathbf{B}_{[i:j]} := (\pi_i(\mathbf{b}_i), \ldots, \pi_i(\mathbf{b}_j))$ is an HKZ reduced basis of $\mathcal{L}(\mathbf{B}_{[i:j]})$ and that $\|\widetilde{\mathbf{b}}_i\| \le \sqrt{j-i+1} \cdot \Pi_{k=i}^{j} \|\widetilde{\mathbf{b}}_k\|^{1/(j-i+1)}$.)

Exercise 2 (Successive Minima)
Let $g(n)$ be the smallest number such that $\lambda_i(\mathcal{L})\lambda_{n-i+1}(\mathcal{L}^*) \le g(n)$, $\forall i \in [n]$ and every n-dimensional lattice $\mathcal{L}$.

Let $\mathcal{L}$ be an n-dimensional lattice. Prove the following statements.

1. $\frac{1}{g(n)} \cdot \frac{\lambda_{n-i+1}(\mathcal{L}^*)}{\lambda_{n-i}(\mathcal{L}^*)} \le \lambda_{i+1}(\mathcal{L})/\lambda_i(\mathcal{L}) \le g(n) \cdot \frac{\lambda_{n-i+1}(\mathcal{L}^*)}{\lambda_{n-i}}, \forall 1 \le i < n$.
2. Assume that $\lambda_{i+1}(\mathcal{L})/\lambda_i(\mathcal{L}) > g(n)$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_i \in \mathcal{L}$ and $\mathbf{v}_1^*, \ldots, \mathbf{v}_{n-i}^* \in \mathcal{L}^*$ respectively denote successive minima of $\mathcal{L}$ and $\mathcal{L}^*$, that is $\|\mathbf{v}_j\| = \lambda_j(\mathcal{L})$, $j \in [i]$, and $\|\mathbf{v}_j^*\| = \lambda_j(\mathcal{L}^*)$, $j \in [n-i]$. Prove that $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_i)$ and $\text{span}(\mathbf{v}_1^*, \ldots, \mathbf{v}_{n-i}^*)$ are orthogonal.
3. Let $W \subseteq \mathbb{R}^n$ be a k-dimensional linear subspace, $k \ge 1$. Assume that $\max_{\mathbf{x} \in W} d(\mathcal{L}, \mathbf{x}) \le R$. Prove that $\lambda_k(\mathcal{L}) \le 2R$.

Exercise 3 (Continuity over the Torus)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying the decay estimate $|f(\mathbf{x})| \le \frac{C}{(1+\|\mathbf{x}\|)^{n+\delta}}$ for some constants $C, \delta > 0$.

Let $\mathcal{L} \subset \mathbb{R}^n$ be a full-rank lattice.

1. Prove that $g(\mathbf{t}) = \sum_{\mathbf{y} \in \mathcal{L}} f(\mathbf{y} + \mathbf{t})$ is a continuous function on $\mathbb{R}^n/\mathcal{L}$.
2. Assume that $\sum_{\mathbf{y} \in \mathcal{L}^*} |\hat{f}(\mathbf{y})| < \infty$. Prove that $h(t) = \sum_{\mathbf{y} \in \mathcal{L}^*} e^{2\pi i \langle \mathbf{y}, \mathbf{t} \rangle} \hat{f}(\mathbf{y})$ is a continuous function on $\mathbb{R}^n/\mathcal{L}$.