1. Let $X_1, \ldots, X_n$ be some random variables on the same probability space. Consider the $n \times n$ covariance matrix $A$ with entries $a_{ij} = E[(X_i - E[X_i])(X_j - E[X_j])]$. Show that any covariance matrix is PSD.

**Solution:** Let us define $Y_i = X_i - E[X_i]$, so that $Y_i$ has mean 0. Then $a_{ij} = E[Y_i Y_j]$. For any $w \in \mathbb{R}^n$ we have

$$w^T A w = \sum_{ij} w_i E[Y_i Y_j] w_j = E\left[\sum_{ij} w_i Y_i w_j Y_j\right] = E\left[\left(\sum_i w_i Y_i\right)^2\right] \geq 0$$

and hence $A$ is PSD.

2. Show that given any PSD matrix $A$, one can construct random variables $X_1, \ldots, X_n$ (in fact jointly Gaussian random variables) with covariance matrix $A$.

**Solution:** Setting $X_i = \langle g, u_i \rangle$, we have

$$E[X_i X_j] = E[\langle g, u_i \rangle \langle g, u_j \rangle] = E\left[\sum_k g(k) u_i(k) \sum_{k'} g(k') u_j(k')\right] = \sum_k u_i(k) u_j(k) = u_i \cdot u_j$$

where the second last step follows as $E[g(k)^2] = 1$ and $E[g(k)g(k')] = 0$ for $k \neq k'$, as the entries of $g$ are iid $N(0, 1)$.

3. Given any graph $G = (V,E)$, construct an explicit solution for the max-cut SDP such that the objective is at least $|E|/2$.

**Solution:** Consider the all-orthogonal solution where vector $i$ is assigned the vector $v_i = e_i$, i.e. unit vector in the $i$-th direction. Then for any two vertices $i$ and $j$, we have $(1/4)||v_i - v_j||^2 = 1/2$, and thus the maxcut SDP objective is $|E|/2$.

4. Here we will show an improved bound of $2/\pi$ for the maximizing the quadratic form $x^T Ay$ where $A$ is a PSD matrix. Note that this is a generalization of the max-cut problem, where $A$ corresponds to the Laplacian of $G$.

We do this in the following steps

- First, show using Cauchy-Schwarz that for any PSD matrix $A$ one has

$$x^T Ay \leq (x^T Ax)^{1/2}(y^T Ay)^{1/2}$$

and hence one can assume in the optimum solution that $x = y$.

**Solution:** As $A$ is PSD, $A = V^T V$ for some matrix $V$. So, $x^T Ay = x^T V^T V y = (V x)^T (V y)$. By Cauchy Schwarz (i.e. $a^T b \leq \|a\|_2 \|b\|_2$ for any two vectors $a$ and $b$) it follows that

$$(V x)^T (V y) \leq \|V x\|_2 \|V y\|_2 = (x^T V^T V x)^{1/2}(y^T V^T V y)^{1/2} = (x^T Ax)^{1/2}(y^T Ay)^{1/2}.$$

- Second, show the following identity. If $b$ and $c$ are two unit vectors, and $g$ is a random gaussian then

$$\frac{\pi}{2} E[\text{sign}(g \cdot b) \cdot \text{sign}(g \cdot c)] = b \cdot c + E\left[|b \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(b \cdot g)| |c \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(c \cdot g)|\right]$$
Hint: To compute $E[(b \cdot G) \cdot \text{sign}(c \cdot G)]$, by rotation invariance assume that $b = (b_1, b_2, 0, \ldots, 0)$ and $c = (1, 0, \ldots, 0)$. This is $E[(b_1g_1 + b_2g_2)(\text{sign}(g_1))] = E[b_1g_1\text{sign}(g_1)]$. Show that this integral is $(\sqrt{2/\pi})b_1$.

**Solution:** Let us expand

$$E \left( b \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(b \cdot g) \right) \left( c \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(c \cdot g) \right)$$

as

$$E[(b \cdot g)(c \cdot g)] - E[(b \cdot g)\sqrt{\frac{\pi}{2}} \text{sign}(c \cdot g)] - E[(c \cdot g)\sqrt{\frac{\pi}{2}} \text{sign}(b \cdot g)] + E[\frac{\pi}{2} \text{sign}(b \cdot g) \text{sign}(c \cdot g)]$$

As we have seen before, the first term is simply $b \cdot c$. Moreover the last term is identical to the lhs of the identity we wish to show. So, it suffices to show that second term $E[(b \cdot g)\sqrt{\frac{\pi}{2}} \text{sign}(c \cdot g)]$ is $b \cdot c$ (similarly for the third term by symmetry). By the hint above, this is the same as $\sqrt{\frac{\pi}{2}} E[b_1g_1\text{sign}(g_1)]$ which we need to show is $b_1$ (which is $b \cdot c$). Now,

$$E[b_1g_1\text{sign}(g_1)] = \int_{-\infty}^{\infty} b_1|x|\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 2b_1 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{-x^2/2} dx$$

$$= 2b_1 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} dy \quad \text{(setting } y = x^2/2)$$

$$= \left(\frac{1}{\pi}\right)^{1/2}b_1$$

Finally, use these two facts (and that $A$ is PSD) to show that the random hyperplane rounding gives at least a $2/\pi$ approximation.

**Solution:** By part 1, we can consider maximizing $x^T Ax$ and consider the vector relaxation $\sum_{ij} a_{ij} u_i u_j$. Let $B$ denote the optimum value of this SDP and let $u_i$ denote some optimum solution.

Applying the hyperplane rounding gives an expected solution value of $\sum_{ij} a_{ij} \text{sign}(\langle u_i, g \rangle) \cdot \text{sign}(\langle u_i, g \rangle)$. Call this value $A$.

Now by the identity in the second part (applying it to each pair $u_i, u_j$, multiplying it by $a_{ij}$ and combining), we have

$$\frac{\pi}{2} A = B + \mathbb{E} \left( \sum_{ij} a_{ij} [u_i \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(u_i \cdot g)] [u_j \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(u_j \cdot g)] \right).$$

Now the key point is that for any value of $g$, the big term on the rhs is non-negative. This follows as $A$ is PSD, and hence $w^T Aw \geq 0$ for any fixed vector $w$. For any fixed $g$, consider the vector $w$ with entry $w_i = u_i \cdot g - \sqrt{\frac{\pi}{2}} \text{sign}(u_i \cdot g)$.

Thus $\frac{\pi}{2} A \geq B$ and we are done.