1. (Variational Characterization of Eigenvalues) Let M be a symmetric real matrix with eigenvaues $\lambda_1 \leq \ldots \leq \lambda_n$. Then

$$\lambda_1 = \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}} \frac{x^T M x}{x^T x}$$

$$\lambda_k = \max_{S: dim(S) = n-k+1} \min_{x \in S - \{\mathbf{0}\}} \frac{x^T M x}{x^T x}$$

$$\lambda_k = \min_{S: dim(S) = k} \max_{x \in S - \{\mathbf{0}\}} \frac{x^T M x}{x^T x}$$

   Hint: Use the spectral decomposition for symmetric matrices that $M = \sum_i \lambda_i u_i u_i^T$ where λ_i are real, and u_i are orthonormal.

2. Let G be a d-regular graph and $L_G = I - A/d$ be the normalized Laplacian of G, Let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of L_G. Then show that
   (a) $\lambda_1 = 0$ and $\lambda_n \leq 2$. The all 1 vector is an eigenvector for λ_1.
   (b) For any integer k, $\lambda_k = 0$ iff G has at least k components.
   (c) $\lambda_n = 2$ iff G has a component that is bipartite.

3. If $X_1, \ldots, X_n$ are random variables taking values in $[0,1]$. Let $\mu_i = \mathbb{E}[X_i]$ and let $\mu = (\sum_i \mu_i)/n$. Then show that at least $\mu/2$ fraction of the random variables have mean at least $\mu/2$.

4. If d and d' are two ℓ_1 metrics on a point set X. Then $d + d'$ is also an ℓ_1 metric.

5. We will show that any ℓ_2 metric can be embedded isometrically into ℓ_1. In particular one can map any point $v \in R^d$ to some $\pi(v)$ so that $\|v - w\|_2 = |\pi(v), \pi(w)|_1$ for every pair of points v, w. Consider the random Gaussian projection $v \to \langle g, v \rangle$ and show why this gives the desired map.

   Hint: Think of one coordinate for each Gaussian. Also, setting $u = v - w$, it suffices to relate $\|u\|_2$ and $E_g[|\langle u, g \rangle|]$.

6. Consider the 4 points $a = (1,1,0,0)$,$b = (0,1,1,0)$, $c = (0,0,1,1)$ and $d = (1,0,0,1)$ in the ℓ_1 metric. So, $d(a,c) = d(b,d) = 2$ and all other distances are 1. Show that they cannot be embedded isometrically into ℓ_2.