1. (Variational Characterization of Eigenvalues) Let $M$ be a symmetric real matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Then

$$
\lambda_1 = \min_{x \in \mathbb{R}^n - \{0\}} \frac{x^T M x}{x^T x} \\
\lambda_k = \max_{S: \dim(S) = n - k + 1} \min_{x \in S - \{0\}} \frac{x^T M x}{x^T x} \\
\lambda_k = \min_{S: \dim(S) = k} \max_{x \in S - \{0\}} \frac{x^T M x}{x^T x}
$$

Hint: Use the spectral decomposition for symmetric matrices that $M = \sum_i \lambda_i u_i u_i^T$ where $\lambda_i$ are real, and $u_i$ are orthonormal.

2. Let $G$ be a $d$-regular graph and $L_G = I - A/d$ be the normalized Laplacian of $G$. Let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of $L_G$. Then show that

(a) $\lambda_1 = 0$ and $\lambda_n \leq 2$. The all 1 vector is an eigenvector for $\lambda_1$.

(b) For any integer $k$, $\lambda_k = 0$ iff $G$ has at least $k$ components.

(c) $\lambda_n = 2$ iff $G$ has a component that is bipartite.

3. If $X_1, \ldots, X_n$ are random variables taking values in $[0,1]$. Let $\mu_i = \mathbb{E}[X_i]$ and let $\mu = (\sum_i \mu_i)/n$. Then show that at least $\mu/2$ fraction of the random variables have mean at least $\mu/2$.

4. If $d$ and $d'$ are two $\ell_1$ metrics on a point set $X$. Then $d + d'$ is also an $\ell_1$ metric.

5. We will show that any $\ell_2$ metric can be embedded isometrically into $\ell_1$. In particular one can map any point $v \in \mathbb{R}^d$ to some $\pi(v)$ so that $\|v - w\|_2 = |\pi(v), \pi(w)|_1$ for every pair of points $v, w$. Consider the random Gaussian projection $v \rightarrow (g, v)$ and show why this gives the desired map.

Hint: Think of one coordinate for each Gaussian. Also, setting $u = v - w$, it suffices to relate $\|u\|_2$ and $E_g[|\langle u, g \rangle|]$.

6. Consider the 4 points $a = (1,1,0,0), b = (0,1,1,0), c = (0,0,1,1)$ and $d = (1,0,0,1)$ in the $\ell_1$ metric. So, $d(a,c) = d(b,d) = 2$ and all other distances are 1. Show that they cannot be embedded isometrically into $\ell_2$. 