1. Let G be a non-empty n vertex graph where vertex $i \in [n]$ has degree $d_i \geq 1$ (no isolated vertices). Define the normalized Laplacian of G by the quadratic form

$$x^T \mathcal{L}_G x = \sum_{(i,j)\in E} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 .$$

   (a) Show that the largest eigenvalue λ_n of $\mathcal{L}_G$ can be expressed as

$$\lambda_n = \max_{\|x\|=1} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i=1}^n d_i x_i^2} .$$

   (b) Prove that $\lambda_n \leq 2$.

   (c) Show that $\lambda_n = 2$ if and only if G is bipartite.

2. Let $T \in \mathbb{R}^{n\times m}$. Assume that we can write $T = AB$, where A has rows of ℓ_2 norm at most R and B has columns of ℓ_2 norm at most R. Prove that T can be expressed as

$$T = \sum_{i=1}^{nm} \alpha_i x_i y_i^T ,$$

   where $x_i \in \{-1,1\}^n, y_i \in \{-1,1\}^m, \alpha_i \geq 0$, $i \in [nm]$, such that

$$\sum_{i=1}^{k} \alpha_i \leq K_G R^2 ,$$

   where K_G is Grothendieck's constant.

3. Show that any ℓ_2^2 metric in d dimensions can contain at most 2^d distinct points.

   [Hint: First try to show this for $d = 2$. In general, for a point p, consider the cone C_p containing the other points and p as the origin.]

4. Let $G = (V,E)$ and $H = (W,F)$ be graphs. Then $G \times H$ is the graph with vertex set $V \times W$ and edge set $((v1,w),(v2,w))$ where $(v_1,v_2) \in E$ and $((v,w1),(v,w2))$ where $(w1,w2) \in F$.

   If $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_m$ be the eigenvalues of the Laplacians of G and H with eigenvectors $x_1,\dots,x_n$ and $y_1,\dots,y_m$ respectively. Then, show that for each $1 \leq i \leq n$ and $1 \leq j \leq m$, $G \times H$ has an eigenvector z of eigenvalue $\lambda_i + \lambda_j$ such that

$$z(v,w) = x_i(v) y_j(v)$$

5. What is the spectrum of the Laplacian of the graph on two vertices consisting of a single edge. Use the above exercise to determine the spectrum of the d-dimensional hypercube. Show that the normalized Laplacian has $\lambda_2 = 2/d$.