Notation: $\mathbb{R}_d[\mathbf{x}]$ corresponds to all polynomials in n variables of degree at most d. We denote the set of SOS polynomials of degree at most d by $\Sigma^2_{n,d} = \left\{\sum_{i=1}^k q_i^2 : q_i \in \mathbb{R}[\mathbf{x}], \deg(q_i) \le d/2\right\}$, which we also denote Σ^2_d when the context is clear. The notation $p \succeq_{\Sigma^2_d} q$ is equivalent to $p - q \in \Sigma^2_d$.

Exercises:

1. Let $p(x) = \sum_{|\alpha|,|\beta|\le d} M_{\alpha,\beta} x^{\alpha+\beta}$ for $M \succeq 0$. Show that $p = 0$ iff $\text{trace}(M) = 0$ (Hint: Use the Cholesky decomposition of M to help point out a non-zero term in p).

2. (Composition rules) Take $p_1, p_2, q_1, q_2 \in \mathbb{R}[\mathbf{x}]_d$. Assume that $p_1^2 \succeq_{\Sigma^2_{2d}} q_1^2$ and $p_2^2 \succeq_{\Sigma^2_{2d}} q_2^2$. Show that then $p_1^2 p_2^2 \succeq_{\Sigma^2_{4d}} q_1^2 q_2^2$.

3. (a) (Motzkin Polynomial) Show that $p(x, y) = x^4y^2 + y^4x^2 + 1 - 3x^2y^2$ is non-negative over $\mathbb{R}[x, y]$ (Hint: use the AM-GM inequality). Prove that p is NOT a sum of squares (Hint: Assume that $p(x, y) = \sum_{i=1}^k q_i(x, y)^2$. Prove that none of the q_i's can have monomials of the form x^i or y^i, $i \in \mathbb{N}$. Conclude that the coefficient of x^2y^2 of the purposed decomposition cannot be -3.)

   (b) Let $L : \mathbb{R}[x]_4 \to \mathbb{R}$ such that $L[1] = 1$, $L[x] = 1$, $L[x^2] = 1$,$L[x^3] = 1$,$L[x^4] = 2$. Show that L is a valid pseudo-expectation operator over $\mathbb{R}$, but that L does not coincide with the moments of any distribution over $\mathbb{R}$.

4. (a) Let $p \in \mathbb{R}[x]$ be a non-negative polynomial over $\mathbb{R}$. Show that p is a sum of squares (Hint: factor p over the complex numbers and combine terms appropriately).

   (b) Show that p above is a sum of exactly two squares. (Hint: use the identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$).

5. Let $p \in \mathbb{R}[x]$ be a convex univariate polynomial over $\mathbb{R}$, $\deg(p) \le d$, and let $L : \mathbb{R}[x]_d \to \mathbb{R}$ be a pseudo-expectation operator.

   (a) Show that for any $t \in \mathbb{R}$, $p(x) - p(t) - p'(t)(x - t) \in \Sigma^2_d$ (Hint: use the previous exercise and convexity of p).

   (b) (Jensen's inequality) Use the above to show that $L[p(x)] \ge p(L[x])$. Conclude that $L[x^{2p}] \ge L[x]^{2p}$ for $p \le d/2$.

   (c) Show that the above extends to showing $L[p(q(x))] \ge p(L[q(x)])$ as long as $L[p(q(x))]$ is defined for L.