Notation: Let $\mathbb{R}[\mathbf{x}]_{H,d} = \mathbb{R}[\mathbf{x}]/I(x_i^2 - x_i : i \in [n])$, i.e. the multilinear polynomials endowed with multiplication rule $pq = \sum_{\alpha,\beta\subseteq[n]} p_\alpha q_\beta x^{\alpha\cup\beta}$. For $p \in \mathbb{R}_H[\mathbf{x}]_d$ let $\deg(p)$ denote the degree of its unique multilinear representative, i.e. $\deg(\sum_{\alpha\subseteq[n]} p_\alpha x^\alpha) = \max\{|\alpha| : \alpha \subseteq [n], p_\alpha \neq 0\}$, and let $\mathbb{R}[\mathbf{x}]_{H,d}$ denote the polynomials in this ring of degree at most d.

Exercises:

1. **MAXCUT:**

   (a) Show that the MAXCUT SDP over an n-vertex graph $G = ([n], E)$ can be expressed as

   $$\begin{aligned} \max \quad & \mathrm{trace}(L_G X) \\ & X_{ii} = 1 \quad \forall i \in [n] \\ & X \succeq 0 \end{aligned} \tag{1}$$

   where $L_G = \sum_{\{i,j\}\in E}(e_i - e_j)^\mathsf{T}(e_i - e_j)$ is the Laplacian of G.
   Let $C = \left\{x_1^2 - 1, \ldots, x_n^2 - 1\right\} \subset \mathbb{R}[\mathbf{x}]_2$ and $m_G(x) = \sum_{\{i,j\}\in E}(x_i - x_j)^2 \in \mathbb{R}[\mathbf{x}]_2$ denote the MAXCUT polynomial. Show that the above program is equivalent to

   $$\max\{L[m_G(x)] : L \in \mathrm{Las}_2(\emptyset, C)\} \ ,$$

   i.e. construct an explicit mapping between solutions.

   (b) Using the above formulation for the MAXCUT SDP show that the dual SDP is

   $$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^n \lambda_i \\ & L_G \preceq \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \ . \end{aligned} \tag{2}$$

   where $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix with λ on the diagonal.
   Next, show that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a solution to the dual SDP if and only if show that

   $$m_G(x) = \sum_{i=1}^n \lambda_i - v(x) + w(x),$$

   where $v \in \Sigma_{n,2}^2$ and $w \in I_2(C)$. Conclude the equivalence of the dual program with the degree 2 sum of squares relaxation for MAXCUT.

   (Hint: What does the feasibility of λ imply about the polynomial $\sum_{i=1}^n \lambda_i x_i^2 - \sum_{\{i,j\}\in E}(x_i - x_j)^2$?)

2. **Stieltjes moment problem:** Let $L \in \mathrm{Las}_{2d}(\{x\}, \emptyset)$ be a pseudo-expectation operator on univariate polynomials of degree at most $2d$. Assume that $L[p(x)^2] > 0$ for all $p \in \mathbb{R}[x]_d$, $p \neq 0$. Show that there exists a discrete measure μ supported on exactly d points on the non-negative axis such that $\mathbb{E}_\mu[q(x)] = L[q(x)]$ for all $q \in \mathbb{R}[x]_{2d}$.

   (Hint: use the proof for the Hamburger moment problem. What does the extra condition $L[xq(x)^2] \geq 0$, $\forall q \in \mathbb{R}_d[x]$, buy you?)

3. **Inclusion-Exclusion:** Prove the formal identity

$$1 = \sum_{I \subseteq [n]} x^I (1_n - x)^{[n] \setminus I} \ ,$$

by expanding out the sum and showing that the coefficients on all the non-trivial monomials cancel.

4. **Degree cancelation:** Find an explicit polynomial $p \in \mathbb{R}[\mathbf{x}]_H$, where $\deg(p) = n$ but $\deg(p^2) = 0$.

(Hint: what are the possible "square roots" of the constant 1 function?)

5. **Partial Integrality of Lasserre on the Hypercube:** Define

$$\mathrm{Las}_{2d}^{H} = \left\{ L : \begin{array}{ll} L : \mathbb{R}[\mathbf{x}]_{H,2d} \to & \text{linear} \\ L[q^2(x)] \geq 0, & \forall q \in \mathbb{R}[\mathbf{x}]_{H,d} \end{array} \right\} . \tag{3}$$

Let $L \in \mathrm{Las}_{2d}^{H}$ and let $I_L = \{i \in [n] : L[x_i] \in \{0,1\}\}$, $R_L = [n] \setminus I_L$ and $P_L : \{0,1\}^n \to \{0,1\}^n$ satisfy

$$P_L(x)_i = \begin{cases} L[x_i] & : i \in I_L \\ x_i & : \text{o/w} \end{cases} .$$

(a) For any $p \in \mathbb{R}[x]_H$ show that

$$p \circ P_L(x) = \sum_{\alpha \subseteq [n]} p_\alpha L[x]^{\alpha \cap I_L} x^{\alpha \setminus I_L} \in \mathbb{R}[x_i : i \in R_L]_H \ ,$$

i.e. the partial evaluation of p with respect to the integral components of the vector $L[x] = (L[x_1], \ldots, L[x_n])$.

(b) Show that for $p \in \mathbb{R}[x]_{H,2d}$ that

$$L[p(x)] = L[p \circ P_L(x)] \ .$$

(Hint: Show that for any monomial x^α, where $i \in I_L \cap \alpha$, that $L[x^\alpha] = L[x_i]L[x^{\alpha \setminus \{i\}}]$, and continue by induction.)

(c) For $p \in \mathbb{R}[\mathbf{x}]_H$ define $\deg_{R_L}(p) = \max\{|R_L \cap \alpha| : \alpha \subseteq [n], p_\alpha \neq 0\}$ and $\mathbb{R}[\mathbf{x}]_{H,2d,R_L} = \{p \in \mathbb{R}[\mathbf{x}]_H : \deg_{R_L}(p) \leq 2d\}$. Define $\hat{L} : \mathbb{R}[\mathbf{x}]_{H,2d,R_L} \to \mathbb{R}$ by the relation

$$\hat{L}[p(x)] = L[p \circ P_L(x)] \ .$$

Show that

$$\hat{L}[q^2(x)] \geq 0 \quad \forall q \in \mathbb{R}[\mathbf{x}]_{H,2d,R_L} \ .$$

(Hint: Show that $q^2 \circ P_L = (q \circ P_L)^2$. What is the degree of $q \circ P_L$?)