Notation: Given a $[0,1]$ polytope $P = \{x \in [0,1]^n : Ax \leq b, Cx = d\}$, $A \in \mathbb{R}^{k \times n}$, $C \in \mathbb{R}^{l \times n}$, for notational convenience we shall define

$$\mathrm{Las}_d^H(P) := \mathrm{Las}_d^H(\{b_i - a_i \cdot x : \forall i \in [k]\}, \{d_j - c_j \cdot x, \forall j \in [l]\}) \ .$$

Exercises:

1. $\{0,1\}$ **vs** $\{-1,1\}$ **Hypercube:** Depending on the problem it is sometimes more convenient to work on the $\{-1,1\}$ hypercube instead of the $\{0,1\}$ hypercube. Here you will explore the basic properties of the $\{-1,1\}$ hypercube, and show that translating problems between the $\{0,1\}$ and $\{-1,1\}$ hypercube can be done completely automatically. Here, we will make crucial use of the invertible linear map $\phi : \{-1,1\}^n \to \{0,1\}^n$ defined by $\phi(x)_i = (x_i+1)/2$, $i \in [n]$.

   Recall that the defining equalities for the $\{-1,1\}$ hypercube are $C = \{x_i^2 - 1 : i \in [n]\}$, i.e. x_i^2 is equivalent to 1, and for the $\{0,1\}$ hypercube they are $H = \{x_i^2 - x_i : i \in [n]\}$. Correspondingly, we define $\mathbb{R}[\mathbf{x}]_C = \mathbb{R}[\mathbf{x}]/I(C)$, and $\mathbb{R}[\mathbf{x}]_{C,d}$ to be the polynomials in this set having degree at most d (with respect to the multilinear representation).

   (a) Show that every polynomial $p \in \mathbb{R}[\mathbf{x}]$ is equivalent to a multilinear polynomial modulo $I(C)$, i.e. every element of $\mathbb{R}[\mathbf{x}]_C$ can be uniquely associated with a multilinear polynomial. Given two mutlinear polynomials p, q, give an explicit formula for the multilinear representative of pq modulo $I(C)$ (multiplication rule). (Hint: set union gets transformed to what on $\{-1,1\}$ hypercube?)

   (b) Show that $p \in I(H)$ if and only if $p(x) = 0 \ \forall x \in \{0,1\}^n$. (Hint: what does the multilinear representation of p look like?). Use the map ϕ to deduce the same for C, namely that $p \in I(C)$ iff $p(x) = 0 \ \forall x \in \{-1,1\}^n$. Finally, conclude that the map $\tau : \mathbb{R}[\mathbf{x}]_H \to \mathbb{R}[\mathbf{x}]_C$ defined by $\tau(p) = p \circ \phi$ is an isomorphism between the quotient rings which preserves degree, i.e. $\deg(\tau(p)) = \deg(p)$.

   (c) For $F_0, G_0 \subseteq \mathbb{R}[\mathbf{x}]_C$ define

   $$\begin{aligned} \mathrm{Las}_d^C(F_0, G_0) = \{L : L : \mathbb{R}[\mathbf{x}]_{C,d} &\to \mathbb{R} \text{ linear}, \\ L[1] &= 1, \\ L[fq^2] &\geq 0, \quad \forall f \in F_0 \cup \{1\}, \deg(f) + 2\deg(q) \leq d, \\ L[gq] &= 0, \quad \forall g \in G_0, \deg(g) + \deg(q) \leq d\} \ . \end{aligned}$$

   Using the above definition, show for that for $p, F, G \subseteq \mathbb{R}[\mathbf{x}]_H$ the problems

   $$\max \left\{ L[p] : L \in \mathrm{Las}_d^H(F, G) \right\}$$

   and

   $$\max \left\{ L[\tau(p)] : L \in \mathrm{Las}_d^C(\tau(F), \tau(G)) \right\}$$

   are equivalent, where τ is defined as above. More precisely, given $L \in \mathrm{Las}_d^H(F,G)$ show that $L' : \mathbb{R}[\mathbf{x}]_{H,d} \to \mathbb{R}$ defined by $L'[q] = L[\tau^{-1}(q)] := L[q \circ \phi^{-1}]$ satisfies $L' \in \mathrm{Las}_d^C(\tau(F), \tau(G))$ and that $L'[\tau(p)] = L[p]$.

(d) **MAXCUT:** Let G be a graph on n vertices. Take $L \in \mathrm{Las}_2^H := \mathrm{Las}_2^H(\emptyset, \emptyset)$. Show that the matrix $X_{ij} = L[(2x_i - 1)(2x_j - 1)]$ is a feasible solution to the MAXCUT SDP and that

$$\mathrm{tr}(XL_G)/4 = L[\sum_{\{i,j\} \in E} (x_i - x_j)^2] ,$$

where L_G is the Laplacian of G.

2. **Improved Decomposition Property for Stable Set:** For a graph G on n vertices, define the stable set polytope as $\mathrm{stab}_G = \{x \in [0,1]^n : x_i + x_j \le 1, \quad \forall \{i,j\} \in E(G)\}$. Let $\alpha(G)$ denote the maximum size of a stable set in G. Show that any $L \in \mathrm{Las}_{2d}^H(\mathrm{stab}_G)$, $d \ge \alpha(G) + 1$, is integral, i.e. that there exists a distribution over independent sets of G that is consistent with L. (Hint: what happens after you condition on $\alpha(G)$ variables being set to 1? Note that the degree is one lower than what you need for the generic decomposition property.)

3. **Scheduling:** Assume we have 2 machines on which we want to schedule n jobs, where each job takes exactly one unit of processing time to complete on either machine (the machines are identical). The main constraint is that the jobs have precedence constraints, i.e. certain jobs must be finished before others can start. We can represent this by a partial order $\prec$, where we say $i \prec j$, if job i must terminate before job j can start (thus if i starts at time period 1, job j can only start during time period 2). Our main goal is to minimize the time T it takes to finish all jobs. We may phrase the problem as to whether all jobs can be scheduled within T time periods using the following time indexed integer program:

$$\begin{array}{lll} \sum_{t=1}^{T} x_{jt} & = 1 \quad \forall j \in [n] & \text{(must schedule each job)} \\ \sum_{j=1}^{n} x_{jt} & \le 2 \quad \forall t \in [T] & \text{(at most 2 jobs per time period)} \\ x_{it'} & \le 1 - x_{jt} \quad \forall i \prec j, t' \ge t & \text{(must schedule } i \text{ before } j) \\ x_{jt} & \in \{0,1\} \quad \forall j \in [n], t \in [T] & \end{array} \tag{1}$$

Our goal will be to analyze the basic properties of Lasserre on this problem. That is, we will analyze $\mathrm{Las}_d^H(P)$, where P corresponds to linear programming relaxation of (1).

(a) Take $L \in \mathrm{Las}_3(P)$. For each job $j \in [n]$, let $C_j = \max\{t \in [T] : L[x_{jt}] > 0\}$ denote the fractional completion time of job j. Show that if $i \prec j$, then $C_i + 1 \le C_j$. (Hint: condition on x_{i,C_i})

(b) Show that the inequalities

$$\sum_{k=1}^{t} x_{ik} \ge \sum_{k=1}^{t+1} x_{jk}, \quad \forall i \prec j, t \in [T-1] . \tag{2}$$

are at least as strong as the inequalities $x_{it'} \le 1 - x_{jt}$ for $i \prec j$, $t' \ge t$. I.e. show that the old inequalities can be obtained from the strengthened ones via linear implications (here you are allowed to use that $0 \le x_{jt} \le 1$ $\forall j, t$ and $\sum_{t=1}^{T} x_{jt} = 1$ $\forall j$).

(c) Take $L \in \mathrm{Las}_6(P)$. Show that L satisfies (2), i.e. show that

$$\sum_{k=1}^{t} L[x_{ik}] \ge \sum_{k=1}^{t+1} L[x_{jk}], \quad \forall i \prec j, t \in [T-1] .$$

(Hint: use the improved decomposition property on the subset of variables $\{x_{lt} : l \in \{i,j\}, t \in [T]\}$.)