Advanced SDPs	**Lecture 8: March 30, 2017**
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In this lecture, we will do an in depth study of conditioning Lasserre on the hypercube. We will then examine a nice application of Lasserre for the Knapsack problem. The main tool we will use for this purpose is the decomposition property, which will help us derive strong integrality information for Lasserre solutions. The material for this lecture is mostly derived from the sources [Rot13, KMN11].

8.1 Notation

For a natural number $d \in \mathbb{N}$, we will use the notation $\lfloor d \rfloor_2 := 2\lfloor d/2 \rfloor$ to denote the largest even integer less than or equal to d. Precisely, $\lfloor d \rfloor_2 = d$ if d is even and $\lfloor d \rfloor_2 = d - 1$ if d is odd.

Definition 1 (Dual Vector Space)**.** *For a real vector space V, we denote the dual vector space $V^* = \{h : V \to \mathbb{R} : h \text{ linear }\}$ to be the set of linear functions from V to $\mathbb{R}$. In particular, a Lasserre operator $L \in \mathrm{Las}_d(F, G)$, is an element of $\mathbb{R}[\mathbf{x}]_d^*$, i.e. the linear functionals from degree d polynomials to the reals.*

8.2 Partial Conditioning

In this section, we will specialize our study of conditioning to the hypercube and understand what can achieve by conditioning a Lasserre solution on a subset of variables.

A useful object when reasoning about partial conditioning, will be a slight extension of the hypercube Lasserre relaxation where we only bound the degree *outside* a subset of variables. For a subset $R \subseteq [n]$, we use the notation $\overline{R} = [n] \setminus R$ to denote the complement of R in $[n]$. For $p \in \mathbb{R}[\mathbf{x}]_H$, define the R degree of p to be $\deg_R(p) = \max\{|\alpha \cap R| : \alpha \subseteq [n], p_\alpha \neq 0\}$ (again, we consider the coefficients of the multilinear representative of p). Similarly, define the out of R degree of p (which will be the somewhat more convenient concept in this lecture), or $\overline{R}$ degree, to be $\deg_{\overline{R}}(p) = \max\{|\alpha \setminus R| : \alpha \subseteq [n], p_\alpha \neq 0\}$. From here, we define $\mathbb{R}[\mathbf{x}]_{H,\overline{R},d}$ to be all polynomials in $\mathbb{R}[\mathbf{x}]_H$ whose out of R degree is at most d. Note that for $S \subseteq R$, we have that $\deg_{\overline{R}}(p) \leq \deg_{\overline{S}}(p)$. Furthermore, note that $\deg(p) = \deg_{\overline{\emptyset}}(p)$ and $\deg_{\overline{[n]}}(p) = 0$ if $p \neq 0$ and $-\infty$ if $p = 0$.

Definition 2 (Degree out of R Lasserre)**.** *Given a polynomial system $F, G \subseteq \mathbb{R}[\mathbf{x}]_H$, $R \subseteq [n]$ and $d \geq 1$, define the Lasserre relaxation of out of R degree d to be*

$$\begin{aligned} \mathrm{Las}^H_{\overline{R},d}(F,G) = \{L \in \ & \mathbb{R}[\mathbf{x}]^*_{H,\overline{R},d} : \\ & L[fq^2] \geq 0, \forall f \in F \cup \{1\}, q \in \mathbb{R}[\mathbf{x}]_H, \deg(f) + 2\deg_{\overline{R}}(q) \leq d, \\ & L[gq] \geq 0, \forall g \in G, q \in \mathbb{R}[\mathbf{x}]_H, \deg(g) + \deg_{\overline{R}}(q) \leq d\} \ . \end{aligned} \tag{8.1}$$

In an analoguous manner we define the truncated out of R degree d sum of squares cone $\Sigma^2_{H,\overline{R},d} = \mathrm{cone}(q^2 : q \in \mathbb{R}[\mathbf{x}]_H, 2\deg_{\overline{R}}(q) \leq d)$, quadratic module $Q_{\overline{R},d}(F) = \mathrm{cone}(fq^2 : f \in F, q \in \mathbb{R}[\mathbf{x}]_H, \deg(f) + 2\deg_{\overline{R}}(q) \leq d)$, ideal $I_{R,d}(G) = \mathrm{span}(gq : g \in G, q \in \mathbb{R}[\mathbf{x}]_H, \deg(g) + \deg_{\overline{R}}(q) \leq d)$.

Notice that we use the actual degrees for the polynomials in F, G when computing degree bounds instead of their out of R degree. As we will see later, this will make these relaxations more amenable to conditioning.

Definition 3 (Operator Restriction)**.** *For $L \in \mathbb{R}[\mathbf{x}]^*_{H,\overline{R},d}$, $S \subseteq R$ and $k \leq d$, define $L_{\downarrow\overline{S},k}$ to be the restriction of L to $\mathbb{R}[\mathbf{x}]_{H,\overline{S},k}$, the polynomials of out of S degree at most k. We shall also write $L_{\downarrow k}$ to denote the restriction to $\mathbb{R}[\mathbf{x}]_{H,k}$, the polynomials of degree at most k. For an operator $L \in (\mathbb{R}[\mathbf{x}]_{H,\overline{R},d})^*$, for notational convenience we shall write $L \in \mathrm{Las}^H_{\overline{S},k}(F,G)$ to mean that $L_{\downarrow\overline{S},k} \in \mathrm{Las}^H_{\overline{S},k}(F,G)$, i.e. that the appropriate restriction is a Lasserre operator.*

The following lemma gives a basic inequality on how much bigger the out of R-degree can be from the out of $R \cup S$-degree.

Lemma 1. *Take $R, S \subseteq [n]$. Then for $p \in \mathbb{R}[\mathbf{x}]_H$, then*

$$\deg_{\overline{R}}(p) \leq \deg_{\overline{R \cup S}}(p) + |S \setminus R| \ .$$

Proof.

$$\begin{aligned} \deg_{\overline{R}}(p) &= \max\{|\alpha \setminus R| : \alpha \subseteq [n], p_\alpha \neq 0\} \\ &\leq \max\{|\alpha \setminus (R \cup S)| + |S \setminus R| : \alpha \subseteq [n], p_\alpha \neq 0\} = \deg_{\overline{R \cup S}}(p) + |S \setminus R| \ . \end{aligned}$$

□

Definition 4 (Partial Evaluation Operator on Hypercube)**.** *Given disjoint subsets $I, J \subseteq [n]$, define the partial evaluation operator $P_{I,J} : \mathbb{R}[\mathbf{x}]_H \to \mathbb{R}[\mathbf{x}]_H$ which replaces x_i by 1 for $i \in I$ and x_j by 0 for $j \in J$. More precisely, for a polynomial $p \in \mathbb{R}[\mathbf{x}]_H$, we have*

$$P_{I,J}(p) = \sum_{\alpha \subseteq [n] \setminus (I \cup J)} x^\alpha \Big(\sum_{H \subseteq I} p_{\alpha \cup H}\Big) \ .$$

For notational convenience, we will often write $q \circ P_{I,J} := P_{I,J}(q)$, since this captures the fact that we are partially evaluating the polynomials with respect to the variables in the $I \cup J$. Note that as an operator on $\mathbb{R}[\mathbf{x}]$, $P_{I,J}$ clearly sends the vanishing ideal of the hypercube $I(x_i^2 - x_i : i \in [n])$ to itself, and hence it is a well-defined linear operator on $\mathbb{R}[\mathbf{x}]_H$. Lastly, clearly $P_{I,J}(P_{I,J}(p)) = P_{I,J}(p)$, i.e. it is a projection operator, and $P_{I,J}(pq) = P_{I,J}(p)P_{I,J}(q)$, i.e. it is a ring homomorphism from $\mathbb{R}[\mathbf{x}]_H$ to itself.

The following technical lemma shows that reweighting by the subset indicator $x^I(1-x)^J$ has the properties one would expect.

Lemma 2. *Let $L \in \mathrm{Las}^H_{R,d}(F, G)$. Let $I, J \subseteq [n]$ be disjoint, $S = I \cup J$ and $k := |S \setminus R| \le d/2$. Then the following holds:*

1. $\deg_{\overline{R}}(px^I(1-x)^J) \le \deg_{\overline{R \cup S}}(p) + k$, $\forall p \in \mathbb{R}[\mathbf{x}]_H$.
2. $L[px^I(1-x)^J] = L[(p \circ P_{I,J})x^I(1-x)^J]$, $\forall p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R \cup S},d-k}$.
3. *If* $L[x^I(1-x)^J] = 0$ *then* $L[px^I(1-x)^J] = 0$, $\forall p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R \cup S},\lfloor d \rfloor_2 - k}$.

Proof. We prove 1. For $p \in \mathbb{R}[\mathbf{x}]_H$, we have that

$$\begin{aligned}
\deg_{\overline{R}}(px^I(1-x)^J) &\le \deg_{\overline{R \cup S}}(px^I(1-x)^J) + |S \setminus R| \quad (\text{ by Lemma 1 }) \\
&\le \deg_{\overline{R \cup S}}(x^I(1-x)^J) + \deg_{\overline{R \cup S}}(p) + |S \setminus R| \quad (\text{ subadditivity of } \deg_{\overline{R \cup S}}) \\
&\le 0 + \deg_{\overline{R \cup S}}(p) + k = \deg_{\overline{R \cup S}}(p) + k \ ,
\end{aligned} \tag{8.2}$$

as needed.

Note that if $\deg_{\overline{R \cup S}}(p) \le d-k$, the above gives $\deg_{\overline{R}}(px^I(1-x)^J) \le \deg_{\overline{R \cup S}}(p)+k \le d$, and hence the expressions in parts 2 and 3 are well-defined.

We prove 2. By linearity, we need only prove this for monomials, so assume that $p = x^K$, where $\deg_{\overline{R \cup S}}(x^K) = |K \setminus (R \cup S)| \le d-k$. We must show that $L[x^K x^I(1-x)^J] = 0$

if $K \cap J \neq \emptyset$ and that $L[x^K x^I (1-x)^J] = L[x^{K\setminus I} x^I (1-x)^J]$ otherwise. Firstly, if $j \in K \cap J$, then

$$\begin{aligned} L[x^K x^I (1-x)^J] &= L[x^{K\setminus\{j\}} x^I (1-x)^{J\setminus\{j\}} x_j (1-x_j)] \\ &= L[x^{K\setminus\{j\}} x^I (1-x)^{J\setminus\{j\}} (x_j - x_j^2)] \\ &= L[0] = 0 \quad \big(\ x_j^2 \equiv x_j \text{ on hypercube }\big)\ . \end{aligned}$$

Now assume that $K \cap J = \emptyset$, then

$$\begin{aligned} L[x^K x^I (1-x)^J] &= L[x^{K\setminus I} (x^{I\cap K})^2 x^{I\setminus K} (1-x)^J] \\ &= L[x^{K\setminus I} x^{I\cap K} x^{I\setminus K} (1-x)^J] \quad \big(\ x_i^2 \equiv x_i \text{ on hypercube }\big) \\ &= L[x^{K\setminus I} x^I (1-x)^J]\ , \end{aligned}$$

as needed.

We prove 3. As before, it suffices to show it when p is a monomial. Assume that $p = x^K$, $K \subseteq [n]$, such that $\deg_{\overline{R \cup S}}(x^K) = |K \setminus (R \cup S)| \leq \lfloor d \rfloor_2 - k$. By part 2, we know that $L[x^K x^I (1-x)^J] = L[P_{I,J}(x^K) x^I (1-x)^J]$. Since $P_{I,J}(x^K) = 0$ if $K \cap J \neq \emptyset$ and $P_{I,J}(x^K) = x^{K\setminus I}$ otherwise, we may assume that K is disjoint from $S = I \cup J$.

Under this assumption, $\deg_{\overline{R \cup S}}(x^K) = |K \setminus (R \cup S)| = |K \setminus R| = \deg_{\overline{R}}(x^K)$. Furthermore, $\deg_{\overline{R}}(x^I (1-x)^J) = |S \setminus R|$. Restating our assumptions, we have $K \cap S = \emptyset$, $|S \setminus R| = k \leq d/2$ and $|K \setminus R| + |S \setminus R| \leq (\lfloor d \rfloor_2 - k) + k = \lfloor d \rfloor_2$. Thus, we can pick $M \subseteq K \setminus R$ which balances the degrees, that is where $|(S \cup M) \setminus R| \leq d/2$ and $|K \setminus (R \cup M)| \leq d/2$ (note that we crucially use that $\lfloor d \rfloor_2$ is even). From here, we may apply Cauchy-Schwarz twice as follows:

$$\begin{aligned} |L[x^I (1-x)^J x^K]| &= |L[(x^I (1-x)^J x^M) x^{K\setminus M}]| \\ &\leq L[(x^I (1-x)^J x^M)^2]^{1/2} L[(x^{K\setminus M})^2]^{1/2} \quad (\text{ Cauchy-Schwarz }) \\ &= L[x^I (1-x)^J x^M]^{1/2} L[x^{K\setminus M}]^{1/2} \quad \big(x_i \equiv x_i^2 \text{ on hypercube }\big) \\ &\leq L[(x^I (1-x)^J)^2]^{1/4} L[(x^M)^2]^{1/4} L[x^{K\setminus M}]^{1/2} \quad (\text{ Cauchy-Schwarz }) \\ &= L[x^I (1-x)^J)]^{1/4} L[x^M]^{1/4} L[x^{K\setminus M}]^{1/2} \,\big(x_i \equiv x_i^2 \text{ on hypercube }\big) \\ &= 0 \quad \big(\text{ by assumption that } L[x^I (1-x)^J] = 0\ \big)\ . \end{aligned}$$

Note that the rebalancing step insures that we only apply Cauchy-Schwarz when both terms in the product have out of R degree at most $d/2$. $\square$

In the next lemma, we give the properties of conditioning with respect to subset of variables. Most importantly, this enforces integrality on the given subset.

Lemma 3 (Subset Conditioning)**.** *Let $L \in \mathrm{Las}^{H}_{R,d}(F, G)$. For disjoint $I, J \subseteq [n]$, $S := I \cup J$, $|S \setminus R| = k$, $k \leq d/2$, then the linear operator*

$$L[p|I,J] = \begin{cases} L[p; x^I(1-x)^J] : & L[x^I(1-x)^J] > 0 \\ 0 : & o/w \end{cases} \quad , \quad \forall p \in \mathbb{R}[\mathbf{x}]_{H,R\cup S,d-k} \ .$$

is well-defined. If $L[x^I(1-x)^J] > 0$, the following holds:

1. **Partial Evaluation on Conditioned Coordinates:**
   $L[p|I,J] = L[p \circ P_{I,J}|I,J]$, $\forall p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R\cup S},d-k}$.
   In particular $L[x_i|I,J] = 1$, $\forall i \in I$, *and* $L[x_j|I,J] = 0$, $\forall j \in J$.

2. **Monotonicity:**
   Assume that $L[x^{I_2}(1-x)^{J_2}] = 0$, *where* $S_2 = I_2 \cup J_2 \subseteq [n]$, I_2, J_2 *disjoint,* $|S_2 \setminus R| \leq d/2$. *Then* $L[x^{I_2}(1-x)^{J_2}|I,J] = 0$.

3. **Lower order Lasserre operator:**
   $L[\cdot|I,J] \in \mathrm{Las}^{H}_{\overline{R\cup S},d-2k}(F,G)$.

Proof. To begin, we first show that $L[\cdot|I,J]$ is well-defined for $p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R\cup S},d-k}$. By Lemma 3 part 1, $\deg_{\overline{R}}(px^I(1-x)^J) \leq \deg_{\overline{R\cup S}}(p) + k \leq d$, and hence $px^I(1-x)^J$ is defined for L. Thus $L[p|I,J] = L[px^I(1-x)^J]/L[x^I(1-x)^J]$ is well-defined when $L[x^I(1-x)^J] > 0$. When $L[x^I(1-x)^J] = 0$, then $L[p|I,J] = 0$ and hence is also well-defined.

Now assume that $L[x^I(1-x)^J] > 0$. We prove part 1. Take $p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R\cup S},d-k}$. Then by Lemma 3 part 2,

$$L[p|I,J] = \frac{L[px^I(1-x)^J]}{L[x^I(1-x)^J]} = \frac{L[(p \circ P_{I,J})x^I(1-x)^J]}{L[x^I(1-x)^J]} = L[p \circ P_{I,J}|I,J] \ .$$

For the in particular, it follows since $x_i \circ P_{I,J} = 1$, for $i \in I$, and $x_j \circ P_{I,J} = 0$, for $j \in J$, and $L[1|I,J] = L[x^I(1-x)^J]/L[x^I(1-x)^J] = 1$.

We prove part 2. First, note that

$$\deg_{\overline{R\cup S_2}}(x^I(1-x)^J) \leq |S \setminus R| \leq \lfloor d \rfloor_2 \leq \lfloor d \rfloor_2 - |S_2| \ ,$$

where the last inequality follows since $|S_2| \leq \lfloor d \rfloor_2$. Therefore, by Lemma 3 part 3, $L[x^{I_2}(1-x)^{J_2}x^I(1-x)^J] = 0 \Rightarrow L[x^{I_2}(1-x)^{J_2}|I,J] = 0$, as needed.

We prove part 3. That is, we will show that $L[\cdot|I,J] \in \mathrm{Las}^H_{\overline{R \cup S}, d-2k}(F,G)$. Take $f \in F \cup \{1\}$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(f) + 2\deg_{\overline{R \cup S}}(q) \leq d - 2k$. From here, by 3 part 1, note that

$$\deg(f) + 2(\deg_{\overline{R}}(qx^I(1-x)^J)) \leq \deg(f) + 2(\deg_{\overline{R \cup S}}(q) + k) \leq d \ .$$

Thus, since $L \in \mathrm{Las}^H_{R,d}(F,G)$, we have that

$$L[fq^2|I,J] = \frac{L[fq^2x^I(1-x)^J]}{L[x^I(1-x)^J]} = \frac{L[f(qx^I(1-x)^J)^2]}{L[x^I(1-x)^J]} \quad \left(\ x_i^2 \equiv x_i \text{ on hypercube }\right)$$
$$\geq 0 \ ,$$

as needed. Take $g \in G$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(g) + \deg_{\overline{R \cup S}}(q) \leq d - 2k$. Then, as above, we have that $\deg(g) + \deg_{\overline{R}}(qx^I(1-x)^J) \leq \deg(g) + \deg_{\overline{R \cup S}}(q) + k \leq d - k \leq d$. Thus, since $L \in \mathrm{Las}^H_{R,d}(F,G)$, we have that

$$L[gq|I,J] = \frac{L[g(qx^I(1-x)^J)]}{L[x^I(1-x)^J]} = 0 \ ,$$

as needed. □

Notice that in the above proof, we crucially used that the way we measure degree of the defining constraints F, G doesn't change as we move from $\mathrm{Las}^H_{R,d}(F,G)$ to $\mathrm{Las}^H_{\overline{R \cup S}, d-2k}(F,G)$. Also, even though $L[\cdot|I,J]$ is a well-defined linear operator on $\mathbb{R}[\mathbf{x}]_{R \cup S, d-k}$, we only get that $L[\cdot|I,J] \in \mathrm{Las}^H_{\overline{R \cup S}, d-2k}(F,G)$, i.e. as a Lasserre operator it loses k extra degrees. This is because of the non-negativity constraints on F, since we need the "extra" k-degrees to express $x^I(1-x)^J$ as a square (note that the equality constraints induced by G in fact do hold at degree $d - k$).

The following lemma shows that at a loss of degree, we can express an operator on lower degree polynomials as a convex combination of operators than are integral on a chosen subset.

Lemma 4 (Partial Conditioning). *$L \in \mathrm{Las}^H_{R,d}(F,G)$ for $R \subseteq [n]$. Then for $S \subseteq R$, $|S \setminus R| = k \leq d/2$, the following holds:*

1. *$(L[x^I(1-x)^{S \setminus I}] : I \subseteq S)$ is a convex combination.*
2. *$\forall p \in \mathbb{R}[\mathbf{x}]_H$ satisfying $\deg_{\overline{R \cup S}}(p) \leq \lfloor d \rfloor_2 - k$, we have that*

$$L[p] = \sum_{I \subseteq S} L[x^I(1-x)^{S \setminus I}] L[p|I, S \setminus I] \ .$$

Proof. We prove 1. Since $\deg_{\overline{R}}(x^I(1-x)^{S\setminus I}) = |S \setminus R| \leq d/2$ and $L \in \mathrm{Las}_{\overline{R},d}(F,G)$, we have that

$$L[x^I(1-x)^{S\setminus I}] = L[(x^I(1-x)^{S\setminus I})^2] \geq 0 \ .$$

Second, by inclusion-exclusion, $1 = \sum_{I\subseteq S} x^I(1-x)^{S\setminus I}$, and hence

$$1 = L[1] = L[\sum_{I\subseteq S} x^I(1-x)^{S\setminus I}] = \sum_{I\subseteq S} L[x^I(1-x)^{S\setminus I}] \ .$$

Thus the vector $(L[x^I(1-x)^{S\setminus I}] : I \subseteq S)$ is a convex combination as required.

We prove 2. Take $p \in \mathbb{R}[\mathbf{x}]_H$ satisfying $\deg_{\overline{R\cup S}}(p) \leq \lfloor d \rfloor_2 - k$. By Lemma 3 part 1, we have that

$$\deg_{\overline{R}}(px^I(1-x)^J) \leq \deg_{\overline{R\cup S}}(p) + k \leq \lfloor d \rfloor_2 \leq d \ ,$$

and hence $px^I(1-x)^J$ is defined for L. Since $1 = \sum_{I\subseteq S} x^I(1-x)^{S\setminus I}$, we thus have that

$$L[p] = L[\sum_{I\subseteq S} x^I(1-x)^{S\setminus I}p] = \sum_{I\subseteq S} L[x^I(1-x)^{S\setminus I}p] \ .$$

By Lemma 3, since $\deg_{\overline{R\cup S}}(p) \leq \lfloor d \rfloor_2 - k$, then $L[x^I(1-x)^{S\setminus I}] = 0 \Rightarrow L[x^I(1-x)^{S\setminus I}p] = 0$. Thus,

$$\begin{aligned} L[p] &= \sum_{I\subseteq S} L[x^I(1-x)^{S\setminus I}p] = \sum_{\substack{I\subseteq S \\ L[x^I(1-x)^{S\setminus I}]>0}} L[x^I(1-x)^{S\setminus I}p] \\ &= \sum_{\substack{I\subseteq S \\ L[x^I(1-x)^{S\setminus I}]>0}} L[x^I(1-x)^{S\setminus I}]L[p|I,J] = \sum_{I\subseteq S} L[x^I(1-x)^{S\setminus I}]L[p|I,J] \ , \end{aligned}$$

as needed. □

In the following lemma shows that if a Lasserre solution is partially integral, then in any evaluation we can replace the corresponding variables by their integer values. Furthermore, the Lasserre operator can be lifted to an operator on a larger set of polynomials at essentially no loss in degree (i.e. we only lose degree if d is odd).

Lemma 5. *Let $L \in \mathrm{Las}^H_{\overline{R},d}(F,G)$ for $d \geq 2$, $I = \{i \in [n] : L[x_i] = 1\}$, $J = \{i \in [n] : L[x_i] = 0\}$ and $S = I \cup J$. Define $H \in \mathbb{R}[\mathbf{x}]^*_{H,\overline{R\cup S},\lfloor d \rfloor_2}$ by $H[p] = L[p \circ P_{I,J}]$. Then the following holds:*

1. $H_{\downarrow\overline{R},\lfloor d \rfloor_2} = L_{\downarrow\overline{R},\lfloor d \rfloor_2}$.

2. $H \in \mathrm{Las}^{H}_{\overline{R \cup S}, \lfloor d \rfloor_2}(F, G)$.

Proof. To begin, we first prove that H is well-defined. Note that for any $p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R\cup S},\lfloor d \rfloor_2}$, the polynomial $p \circ P_{I,J}$ only contains monomials with support in $[n] \setminus S$. Given this, we have that

$$\deg_{\overline{R}}(p \circ P_{I,J}) = \deg_{\overline{R \cup S}}(p \circ P_{I,J}) \leq \deg_{\overline{R\cup S}}(p) \leq \lfloor d \rfloor_2 \ ,$$

and hence $H[p] = L[p \circ P_{I,J}]$ is well-defined, as needed.

We now prove part 1. As usual, by linearity it suffices to show $L[p] = H[p]$ when p is a monomial. Let $p = x^K$, where $|K \setminus R| \leq \lfloor d \rfloor_2$. Note that $x^K \circ P_{I,J} = 0$ if $K \cap J \neq \emptyset$ and $x^K \circ P_{I,J} = x^{K \setminus I}$ otherwise. To get the desired equality, we must thus show $L[x^K] = 0$ if $K \cap J \neq \emptyset$ and $L[x^K] = L[x^{K\setminus I}]$ otherwise.

Assume first that $\exists j \in K \cap J$. Since $L[x^j] = 0$, $\deg_{\overline{R}}(x^j) \leq 1 \leq d/2$, and $\deg_{\overline{R}}[x^{K\setminus\{j\}}] + \deg_{\overline{R}}[x^j] = \deg_{\overline{R}}[x^K] \leq \lfloor d \rfloor_2$, by Lemma 3 we have that $L[x^K] = 0$ as needed.

Now assume that $K \cap J = \emptyset$. We prove that $L[x^K] = L[x^{K\setminus I}]$ by induction on $|I \cap K|$. The base case $|I \cap K| = 0$ trivially holds, so assume that $|I \cap K| \geq 1$. Pick $i \in I \cap K$. We will show that $L[x^K] = L[x^{K\setminus\{i\}}] \Leftrightarrow L[x^{K\setminus\{i\}}(1 - x_i)] = 0$. Since $L[1 - x_i] = 0$, $\deg_{\overline{R}}(1 - x_i) \leq 1 \leq d/2$ and $\deg_{\overline{R}}(x^{K\setminus\{i\}}) + \deg_{\overline{R}}(x^i) = \deg_{\overline{R}}(x^K) \leq \lfloor d \rfloor_2$, by Lemma 3 we have that $L[x^K] = L[x^{K\setminus\{i\}}]$. Applying the induction hypothesis on $x^{K\setminus i}$, noting that $|(K \setminus \{i\}) \cap I| = |K \cap I| - 1$, we get that $L[x^K] = L[x^{K\setminus\{i\}}] = L[x^{K\setminus I}]$, as needed.

We now prove part 2. Take $f \in F \cup \{1\}$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(f) + 2\deg_{\overline{R\cup S}}(q) \leq \lfloor d \rfloor_2$. We must show that $H[fq^2] \geq 0$. As shown in the beginning of the Lemma, $\deg_{\overline{R}}(q \circ P_{I,J}) \leq \deg_{\overline{R\cup S}}(q)$ and hence $\deg(f) + 2\deg_{\overline{R}}(q \circ P_{I,J}) \leq \lfloor d \rfloor_2$. Since

$$P_{I,J}(fq^2) = P_{I,J}(f)P_{I,J}(q)^2 = P_{I,J}(fP_{I,J}(q)^2),$$

by part 1 we have that

$$H[fq^2] := L[P_{I,J}(fq^2)] = L[P_{I,J}(fP_{I,J}(q)^2)] = L[fP_{I,J}(q)^2] \geq 0,$$

where the last equality and inequality hold since $\deg(f) + 2\deg_{\overline{R}}(q \circ P_{I,J}) \leq \lfloor d \rfloor_2 \leq d$.

Now take $g \in F$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(g) + 2\deg_{\overline{R\cup S}}(q) \leq \lfloor d \rfloor_2$. As above, note that $\deg(g) + 2\deg_{\overline{R}}(q \circ P_{I,J}) \leq \lfloor d \rfloor_2$, and hence

$$H[gq] := L[P_{I,J}(gq)] = L[P_{I,J}(gP_{I,J}(q))] = L[gP_{I,J}(q)] = 0 \ ,$$

where the last two equalities follow since $\deg(g) + 2\deg_{\overline{R}}(q \circ P_{I,J}) \leq \lfloor d \rfloor_2 \leq d$. $\square$

8.3 The Decomposition Property

Definition 5 (Low Degree Projection)**.** *For $S \subseteq [n]$, $k \leq n$, define the projection $\pi_{S,k} : \mathbb{R}[\mathbf{x}]_H \to \mathbb{R}[\mathbf{x}]_{H,S,k}$ by $\pi_{S,k}(p) = \sum_{\alpha \subseteq [n], |\alpha \cap S| \leq k} p_\alpha x^\alpha$. In words, $\pi_{S,k}$ projects out all monomials of S-degree greater than k. Similarly, $\pi_{\overline{S},k}$ projects out all monomials of out of S-degree greater than k.*

The following lemma tells gives a sufficient condition for being able to lift a Lasserre solution to a high degree solution on a subset of variables, and at small cost loss of degree on the remaining variables.

Lemma 6 (Flat Extension of Hypercube Lasserre)**.** *Let $L \in \mathrm{Las}^H_{\overline{R},d}(F,G)$. For $S \subseteq [n]$ and k, $1 \leq k \leq d/2$, define the linear operator $H \in \mathbb{R}[\mathbf{x}]^*_{H,\overline{R \cup S},d-k}$ by $H[p] := L[\pi_{S,k}(p)]$. Then the following holds:*

1. *If $\forall I \subseteq [n]$, $|I \cap S| \geq k+1$, $|I \setminus R| \leq d$, we have $L[x^I] = 0$, then $H \in \mathrm{Las}^H_{\overline{R \cup S},d-2k}(F,G)$.*
2. *If $k+1 \leq d/2$ and $\forall I \subseteq S$, $|I| = k+1$, we have $L[x^I] = 0$, then $H \in \mathrm{Las}^H_{\overline{R \cup S},\lfloor d \rfloor_2 - 2k}(F,G)$.*

Furthermore, in both cases, for all $p \in \mathbb{R}[\mathbf{x}]_H$ satisfying $\deg_{\overline{R}}(p) \leq d$ and $\deg_{\overline{R \cup S}}(p) \leq \lfloor d \rfloor_2 - k$, we have that

$$L[p] = \sum_{I \subseteq S, |I| \leq k} H[x^I(1-x)^{S \setminus I}] H[p | I, S \setminus I] ,$$

where $H[p|I,S \setminus I] \in \mathrm{Las}^H_{\overline{R \cup S},\lfloor d \rfloor_2 - 2k}(F,G)$, for any $I \subseteq S$ such that $H[x^I(1-x)^{S \setminus I}] > 0$.

Proof. To begin, we first show that H is well-defined on $\mathbb{R}[\mathbf{x}]_{\overline{R \cup S},d-k}$. More generally, we show that for any $p \in \mathbb{R}[\mathbf{x}]_H$

$$\deg_{\overline{R}}(\pi_{S,k}(p)) \leq \deg_{\overline{R \cup S}}(p) + k \tag{8.3}$$

As usual, it suffices to show this for monomials. Let $p = x^I$ for some $I \subseteq [n]$. First, since $\pi_{S,k}(x^I) = 0$ if $|I \cap S| \geq k+1$, we may assume that $|I \cap S| \leq k$ and hence that $\pi_{S,k}(x^I) = x^I$. From here, note that

$$\deg_{\overline{R}}(x^I) = |I \setminus R| \leq |I \setminus (R \cup S)| + |S \cap I| \leq \deg_{\overline{R \cup S}}(x^I) + k ,$$

as needed. From here, for any $p \in \mathbb{R}[\mathbf{x}]_{\overline{R \cup S}, d-2k}$, by (8.3), we have that $\deg_{\overline{R}}(\pi_{S,k}(p)) \leq d - k + k = d$, and hence is defined for L, and hence H is defined for p.

We now prove 1. Firstly, note that for $p \in \mathbb{R}[\mathbf{x}]_{H,\overline{R},d}$ we clearly have that $H[p] = L[\pi_{S,k}(p)] = L[p]$, since by assumption, L sends every monomial of S-degree greater than k and out of R degree at most d to 0. Now take $f \in F \cup \{1\}$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(f) + 2\deg_{\overline{R \cup S}}(q) \leq d - 2k$. We must show that $H[fq^2] \geq 0$.

Now by (8.3), we see that

$$\deg(f) + 2\deg_{\overline{R}}(\pi_{S,k}(q)) \leq \deg(f) + 2(\deg_{\overline{R \cup S}}(q) + k) \leq d \ ,$$

and hence since $L \in \mathrm{Las}^{H}_{\overline{R},d}(F, G)$, we have that

$$L[f\pi_{S,k}(q)^2] \geq 0 \ .$$

From here, the desired inequality follows by noting that

$$H[fq^2] = H[f(\pi_{S,k}(q))^2] = L[f(\pi_{S,k}(q))^2] \geq 0 \ .$$

We now check the equality constraints. Take $g \in G$ and $q \in \mathbb{R}[\mathbf{x}]_H$ such that $\deg(g) + \deg_{\overline{R \cup S}}(q) \leq d - 2k$. Similarly to the previous case, we have that $\deg(g) + \deg_{\overline{R}}(\pi_{S,k}(q)) \leq \deg(g) + \deg_{\overline{R \cup S}}(q) + k \leq d - k \leq d$ and hence

$$L[g\pi_{S,k}(q)] = 0 \ .$$

The desired equality follows by noting that

$$H[gq] = H[g\pi_{S,k}(q)] = L[g\pi_{S,k}(q)] = 0 \ ,$$

as needed.

We now prove 2. To begin, we show that $L[x^J] = 0$ for all $J \subseteq [n]$, $|J \cap S| \geq k+1$, $|J \setminus R| \leq \lfloor d \rfloor_2$. Since we can write $J = I \cup (J \setminus I)$ such that $I \subseteq S$, $|I| = k+1 \leq d/2$ and $|J| \leq \lfloor d \rfloor_2$, by Lemma 3 we have that $L[x^J] = 0$ as needed. We now conclude the proof of part 2 by applying part 1 to the operator $L_{\downarrow \overline{R}, \lfloor d \rfloor_2}$, noting that the desired conditions hold for the restricted operator.

We now prove the furthermore. Note that for $p \in \mathbb{R}[\mathbf{x}]_H$ satisfying $\deg_{\overline{R}}(p) \leq d$ and $\deg_{\overline{R \cup S}}(p) \leq \lfloor d \rfloor_2 - k$, we have that both L and H are defined for p and hence $L[p] = H[p]$. From here, applying Lemma 4 part 2 to H and subset S (noting that in this circumstance S has effective degree 0 with respect to H) we get that

$$\begin{aligned} L[p] = H[p] &= \sum_{I \subseteq S} H[x^I(1-x)^{S \setminus I}] H[p | I, S \setminus I] \\ &= \sum_{I \subseteq S, |I| \leq k} H[x^I(1-x)^{S \setminus I}] H[p | I, S \setminus I] \ , \end{aligned}$$

where the last equality follows since $H[x^I(1-x)^{S\setminus I}] = 0$ by construction $\forall I \subseteq S$ such that $|I| > k$.The fact that $H[\cdot|I, S\setminus I] \in \mathrm{Las}^{H}_{\overline{R\cup S},\lfloor d\rfloor_2 - 2k}(F,G)$, when $I \subseteq S$ and $L[x^I(1-x)^{S\setminus I}] > 0$, follows directly from Lemma 3 part 3 applied to the operator H and the subsets I and $S\setminus I$. This concludes the proof. □

Most combinatorial problems can be expressed as optimization over integer points satisfying linear constraints. We shall therefore use slightly more compact notation to describe Lasserre over the hypercube with additional linear constraints.

Definition 6 (Lasserre on a Polytope)**.** *For a polytope* $P = \{x \in \mathbb{R}^n : Ax \le b, Cx = d\}$, $A \in \mathbb{R}^{k\times n}$, $C \in \mathbb{R}^{l\times n}$, $R \subseteq [n]$, $d \in \mathbb{N}$, *define* $\mathrm{Las}^{H}_{\bar{R},d}(P) := \mathrm{Las}^{H}_{\bar{R},d}(\{b_i - a_i\cdot x : i \in [k]\}, \{d_j - c_j \cdot x : j \in [l]\})$*. We similarly define* $\mathrm{Las}_d(P)$*, for general degree* d *Lasserre (not necessarily on the hypercube).*

Exercise 1. *Show that given two equivalent representations of a polytope* $P = \{x \in \mathbb{R}^n : A^1x \le b^1, C^1x = d\} = \{x \in \mathbb{R}^n : A^2x \le b^2, C^2x = b^2\}$ *that the corresponding Lasserre relaxation is identical. Namely, for any* $d \ge 1$*, show that*

$$\mathrm{Las}^{H}_{\bar{R},d}(\left\{b^1_i - a^1_i\cdot x : i \in [k_1]\right\}, \left\{d^1_j - c^1_j\cdot x : j \in [l_1]\right\}) =$$
$$\mathrm{Las}^{H}_{\bar{R},d}(\left\{b^2_i - a^2_i\cdot x : i \in [k_2]\right\}, \left\{d^2_j - c^2_j\cdot x : j \in [l_2]\right\})\ .$$

Hint: use Farkas lemma.

For many problems combinatorial problems, there are natural bounds on the support of any solution, which hold even for the basis LP relaxations. We will use this these properties for establishing conditions under which the flat extension lemma above can be applied.

Definition 7 (Integral Ones Property)**.** *For a polytope* $P \subseteq \mathbb{R}^n$ *and subset* $S \subseteq [n]$*, define*

$$\mathrm{ones}_P(S) = \{\max|\{i \in S : x_i = 1\}| : x \in P, x \in [0,1]^n\}\ .$$

We say that P *satisfies the integral ones property on* S*, if every point* $x \in P \cap [0,1]^n$ *satisfying* $|\{i \in S : x_i = 1\}| = \mathrm{ones}_P(S)$ *is integral on* S*, i.e. satisfies* $x_i \in \{0,1\}$ $\forall i \in S$.

A special property of Lasserre over an LP relaxation of a combinatorial problem, is that the degree 1 moment vector

$$L[\mathbf{x}] := (L[x_1], \ldots, L[x_1]),$$

always yield a feasible solution to the LP relaxation.

Lemma 7 (LP feasibility)**.** *Let $P = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\} \subseteq \mathbb{R}^n$ be a polytope and let $L \in \mathrm{Las}_d(P)$ (or $\mathrm{Las}_d^H(P)$) for $d \geq 1$. Then $L[\mathbf{x}] \in P$.*

Proof. Let $f_i(x) = b_i - a_i \cdot x$, $i \in [k]$, and $g_j(x) = d_j - c_j \cdot x$, $j \in [l]$. Since $f_1, \dots, f_k$ and $g_1, \dots, g_l$ are the defining inequalities and have degree ≤ 1, we have that

$$0 \leq L[f_i] = b_i - \sum_{s=1}^{n} a_{is} L[x_s] = b_i - a_i \cdot L[\mathbf{x}] \quad \forall i \in [k] \ .$$
$$0 = L[g_j] = d_j - \sum_{s=1}^{n} a_{js} L[x_s] = d_j - c_j \cdot L[\mathbf{x}] \quad \forall j \in [l] \ .$$

Thus $L[\mathbf{x}] \in P$ as needed. □

We now use the integral ones property and the fact above to derive a very useful partial integrality condition for Lasserre known as the decomposition property.

Theorem 1 (Decomposition Property)**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope, $R, S \subseteq [n]$, $k := \mathrm{ones}_P(S)$, $d \geq 1$, $L \in \mathrm{Las}_{R,d}^{H}(P)$ and let $H \in \mathbb{R}[\mathbf{x}]^*_{H,\overline{R \cup S},d-k}$ be defined by $H[p] = L[\pi_{S,k}(p)]$. If $2k+3 \leq d$, or, $2k+1 \leq d$ and P satisfies the integral ones property on S, the following holds:*

1. *$L[x^I] = 0$, $\forall I \subseteq S$, $|I| = k+1$, $L[x^I] = 0$.*
2. *For $p \in \mathbb{R}[\mathbf{x}]_H$, $\deg_{\overline{R}}(p) \leq d$ and $\deg_{\overline{R \cup S}}(p) \leq \lfloor d \rfloor_2 - k$, we have that*

$$L[p] = \sum_{I \subseteq S, |I| \leq k} H[x^I (1-x)^{S \setminus I}] H[p | I, S \setminus I]$$

where $H[\cdot | I, S \setminus I] \in \mathrm{Las}^H_{\overline{R \cup S}, d-2k}(P)$, for all $I \subseteq S$ satisfying $H[x^I(1-x)^{S \setminus I}] > 0$. In particular, using at most $O(|S|^k)$ evaluations of H, one can compute $I \subseteq S$, $|I| \leq k$, such that $L[p] \leq H[p | I, S \setminus I]$.

Proof. We prove 1. For the sake of contradiction, let us assume that there exists $I \subseteq S$, $|I| = k+1$ such that $L[x^I] > 0$. Let us first assume that $2k+3 \leq d$. Then by Lemma 3, we have that $L[\cdot | I, \emptyset] \in \mathrm{Las}^H_{R \cup I, d-2(k+1)}(P)$. Thus, since by assumption $d - 2(k+1) \geq 1$, by Lemma 7 we have that $L[\mathbf{x} | I, \emptyset] \in P$. However, $L[x_i | I, \emptyset] = 1$ $\forall\ i \in I \subseteq S$, but this implies that $\mathrm{ones}_P(S) \geq |I| = k+1$, a clear contradiction.

Now assume that $2k+1 \leq d$ and that P satisfies the integral ones property on S. First if $k = 0$, then $L[\mathbf{x}] \in P$ and hence by the integral ones property $L[x_i] = 0$, $\forall i \in S$, as needed. Next if $k \geq 1$, note that $2k+1 \leq d \Rightarrow k+1 \leq \lfloor d \rfloor_2$. Now pick $i^* \in I$ and let $I = I \setminus \{i^*\}$. By Lemma 3 part 3, since $|J| = k \leq d/2$ and $|I| = k+1 \leq \lfloor d \rfloor_2$ and $L[x^I] > 0$, we must also have $L[x^J] > 0$. Thus by Lemma 3, $L[\cdot|J, \emptyset] \in \text{Las}^H_{R\cup J, d-2k}(P)$. Since $d - 2k \geq 1$, by Lemma 7 we that have $L[\mathbf{x}|J] \in P$. By integral ones property on $S \supseteq J$, since $L[x_j|J] = 1$, $\forall j \in J$ and $|J| = k$, we must have that $L[x_i|J] = 0$ for $i \in S \setminus J$ (otherwise we would have too many ones). In particular, $L[x_{i^*}|J] = 0$, a contradiction since $L[x_{i^*}|J] := L[x^J x_{i^*}]/L[x^J] = L[x^I]/L[x^J] > 0$ by our initial assumption.

We now prove 2. The first part follows directly from Lemma 6 part 1 combined with part 1 of the theorem. For the second part, it follows directly from the fact that the vector $(H[x^I(1-x)^{S\setminus I}] : I \subseteq S, |I| \leq k)$ corresponds to a convex combination. Thus, we can find the desired conditioning subset I by trying all $\sum_{i=0}^{k} \binom{|S|}{i} = O(|S|^k)$ possibilities. $\square$

8.3.1 Application to the Knapsack Problem

Given weights $w_1, \ldots, w_n \in [0,1]$ and profits $p_1, \ldots, p_n \geq 0$ for n items, the knapsack problem is to find a maximum profit subset of the items that fits with the knapsack, namely

$$\text{OPT} := \max \left\{ \sum_{i=1}^{n} p_i x_i : x \in \{0,1\}^n, \sum_{i=1}^{n} w_i x_i \leq 1 \right\} .$$

Here the relevant polytope $P = \{x \in \mathbb{R}^n : \sum_{i=1}^n w_i x_i \leq 1\}$ is the knapsack capacity constraint. We will now use the decomposition property to bound the integrality gap of the Lasserre relaxation over the knapsack polytope.

Theorem 2. *[KMN11] For $d \geq 2t+3$,*

$$(1 - 1/t) \max \left\{ L[\sum_{i=1}^{n} p_i x_i] : L \in \text{Las}_d(P) \right\} \leq OPT .$$

Proof. Let $L \in \text{Las}_d(P)$ denote an optimal Lasserre solution. Define $S = \{i \in [n] : p_i \geq \text{OPT}/t\}$ and let $k = \text{ones}_P(S)$. Clearly, $k \leq t$, since otherwise by dropping all the fractionally picked items from the solution in $P \cap [0,1]^n$ achieving the maximum number of ones in S we would get an integral solution of value greater than OPT. Therefore by the decomposition property, there exists $H \in \text{Las}_{d-2k}(P)$, $d - 2k \geq 3$, such that $H[x_i] \in \{0,1\}$, $\forall i \in S$, and for which $L[\sum_{i=1}^n x_i p_i] \leq H[\sum_{i=1}^n x_i p_i]$ and $H[\mathbf{x}] \in P \cap [0,1]^n$. Let $F = \{i \in [n] : H[x_i] \in (0,1)\}$ denote the set of items fractionally picked by H, where we note that $S \cap F = \emptyset$. Let $y_i = H[x_i]$, for $i \in S$, and let $(y_j : j \in F)$ denote the greedy solution for the restricted knapsack on the fractional items with capacity $1 - \sum_{i \in S} y_i$. That is, letting $\pi : [|F|] \to F$ denote the permutation which orders the fractional items according to their bang per buck, namely such that $p_{\pi[i]}/w_{\pi[i]} \geq p_{\pi[i+1]}/w_{\pi[i+1]}$, $\forall i \in [|F|-1]$, the solution y on F picks items integrally in this order until just before the capacity of the knapsack is exceeded. At this point, there is only enough space in the knapsack to pick the next item in this order fractionally, which y does until the knapsack is completely filled. From here, it is easy to check that y achieves greater profit that H, i.e. $\sum_{i=1}^n p_i y_i \geq H[\sum_{i=1}^n p_i x_i]$, since the greedy yields the optimal LP solution for the knapsack restricted to F. In particular, by construction, we also get that $\sum_{i=1}^n p_i y_i \geq \sum_{i=1}^n L[x_i] p_i \geq \mathrm{OPT}$. Note that if y is integral, we are done, since then also $\sum_{i=1}^n p_i y_i \leq \mathrm{OPT}$. Otherwise, note that y has exactly one fractional coordinate y_{i^*}, for $i^* \in F$. Since $i^* \notin S$ by construction, note that $p_{i^*} < \mathrm{OPT}/t$. In particular, we get that OPT has value at least that of the integral solution obtained by dropping i^* from y, and hence

$$\begin{aligned}
\mathrm{OPT} \geq \sum_{i \in [n], i \neq i^*} p_i y_i \geq \sum_{i=1}^n p_i y_i - p_i^* \geq \sum_{i=1}^n p_i y_i - \mathrm{OPT}/t \\
\geq (1 - 1/t) \sum_{i=1}^n y_i p_i \geq (1 - 1/t) \sum_{i=1}^n p_i L[x_i] \ ,
\end{aligned}$$

as needed. □

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