Diving for Conflicts in Mixed-Integer Programming

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Question

Primal heuristics

- incomplete methods without guarantees
- e.g., rounding, diving, LNS, ...
- goal: find near-optimal solutions

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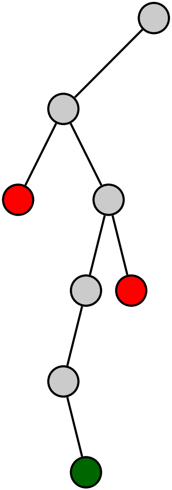
Conflict information

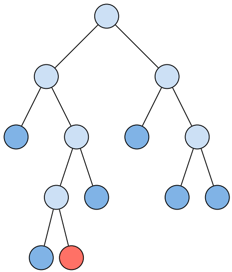
- information on infeasible subproblems
- e.g. conflict analysis, dual ray analysis, clique table, ...
- goal: find valid reductions



Introduction: Conflict Analysis

Goal: When branch-and-bound reaches an infeasible subproblem, analyze the infeasibility to

- extract a shorter reason



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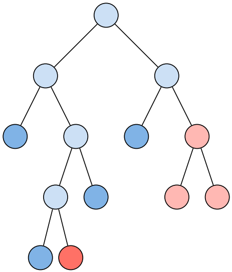
- that prunes other parts of the tree



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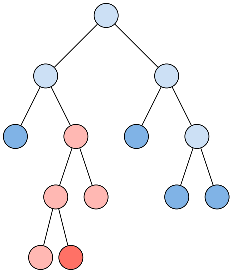

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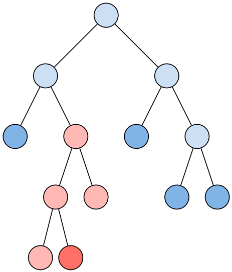

Example: contradicting bound changes after propagation

$$x_1 + x_2 + 2x_3 \leq 2 \qquad (1)$$

$$x_1 + x_2 - 2x_3 \leq 0 \qquad (2)$$

$$x_1, x_2, x_3 \in \{0, 1\}$$

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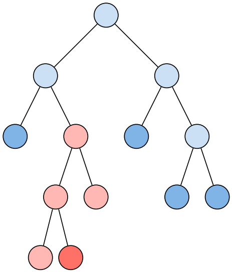

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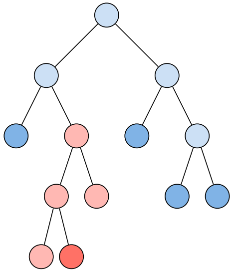

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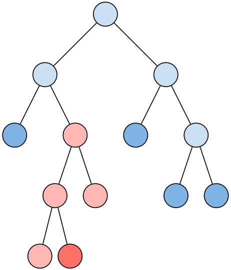

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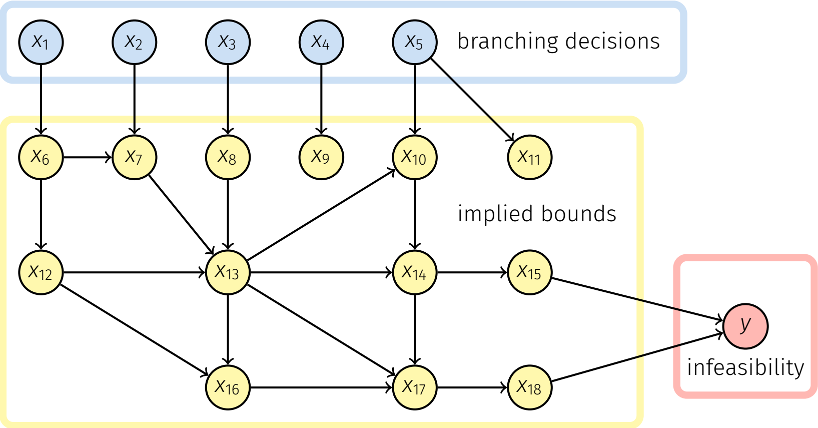
$\rightsquigarrow$ $x_2 \geq 1$ is a sufficient reason

Conflict Graph Analysis

- Goes back to Marques-Silva and Sakallah (1999)
- Idea: consider implications that led to the local bounds



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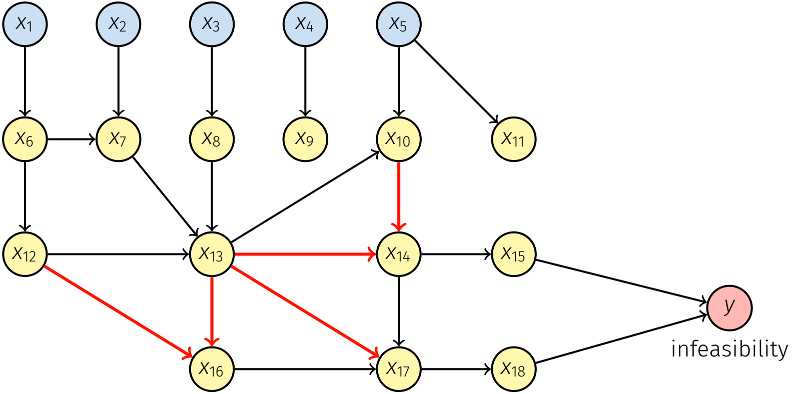
- Each cut that separates branching nodes from y yields a conflict constraint



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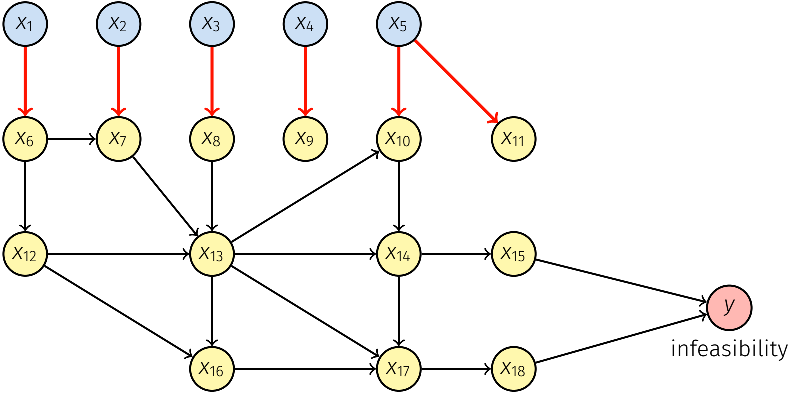
- Separating only branching nodes ⇒ no-good cut



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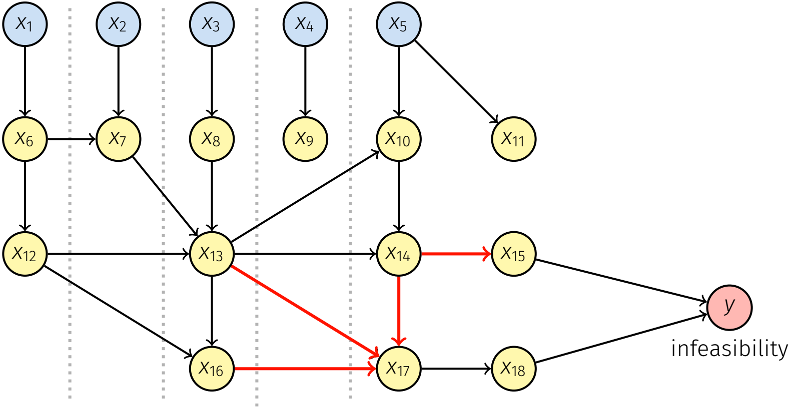
- better: First Unique Implication Point (FUIP), ...



LP Infeasibility

- Assume a subproblem with bounds $\ell \leq \ell' \leq u' \leq u$

$$\min\{c^T x \mid Ax \geq b,\ \ell' \leq x \leq u',\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}\} \tag{3}$$

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- LP of (3) infeasible $\iff$ unbounded direction in the dual

$$\max\{y^T b + r^T\{\ell', u'\} \mid y^T A + r^T = c^T,\ y \in \mathbb{R}^m_+, r \in \mathbb{R}^n\} \tag{4}$$

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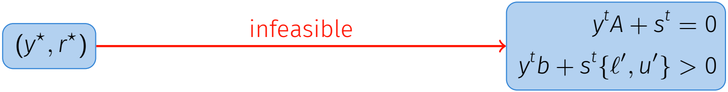
- Farkas constraint globally valid: propagate during tree search
- analogous extension for bound exceeding LPs

Motivation

Starting point of a diving heuristic

- primal feasible $x^\star$ for $\min\{c^t x \mid Ax \geq b, \ell' \leq x \leq u'\}$
- dual feasible $(y^\star, r^\star)$ for $\max\{y^t b + r\{\ell', u'\} \mid y^t A + r = c\}$



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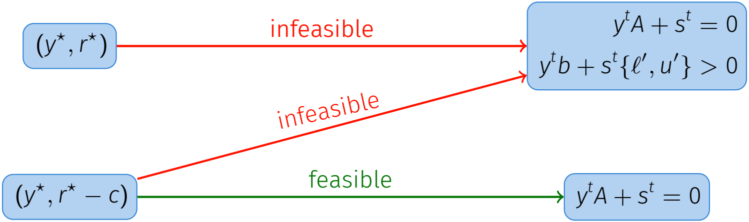

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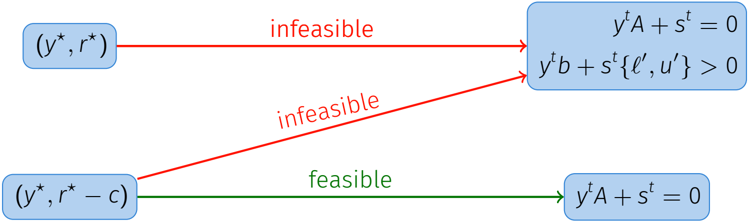

Question: How to satisfy $y^t b + s^t\{\ell', u'\} > 0$?

Motivation

$$(y^\star)^t b + (r^\star - c)^t\{\ell', u'\} > 0$$

$$\iff \quad (y^\star)^t b + \sum_{(r_i^\star - c_i) > 0} (r_i^\star - c_i) \cdot \ell_i' + \sum_{(r_i^\star - c_i) < 0} (r_i^\star - c_i) \cdot u_i' > 0$$

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$$(y^\star)^t b + (r^\star - c)^t\{\ell', u'\} > 0$$

$$\iff \quad (y^\star)^t b + \sum_{(r_i^\star - c_i)>0} (r_i^\star - c_i) \cdot \ell'_i + \sum_{(r_i^\star - c_i)<0} (r_i^\star - c_i) \cdot u'_i > 0$$

Violation reduced by

- **increasing** ℓ'_i if $r_i^\star - c_i > 0$
- **decreasing** u'_i if $r_i^\star - c_i < 0$

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Observation

1. Each fractional candidate i with $x_i^\star \notin \mathbb{Z}$ is basic.
2. Basic variables have reduced cost $r_i^\star = 0$.

$\rightsquigarrow$ Dive optimistically: towards best bound / pseudo solution.

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Violation reduced by

increasing ℓ_i' if $c_i < 0$

decreasing u_i' if $c_i > 0$

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Candidate Selection

Dual view: `fd-dual`

- Goal: reduce violation of $y^t b + s^t\{\ell', u'\} > 0$
- Choose candidate with maximal $c_i \cdot (\lceil x_i^\star \rceil - \ell')$ resp. $c_i \cdot (u_i' - \lfloor x_i^\star \rfloor)$

Primal view: `fd-primal`

- Goal: increase objective function value
- Choose candidate with maximal $c_i \cdot (1 - f_i)$ resp. $c_i \cdot f_i$, where $f_i := x_i^\star - \lfloor x_i^\star \rfloor$ is the fractionality in the LP solution x^*

Summary

1. Learning from infeasibility
   - classical conflict analysis and LP extensions
2. Farkas diving
   - optimistic diving towards best bound relaxation
   - "diving for conflicts"
3. Outlook: "conflicts for diving"?
   - traditional coefficient diving: dive towards few "variable locks" from model constraints
   - conflicts diving: dive towards few "conflict locks" from Farkas constraints

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Thank you for your attention!