Discrepancy of Random Set Systems

Rebecca Hoberg and Thomas Rothvoß

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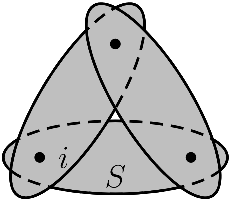
Discrepancy theory

- Set system with m sets, n elements



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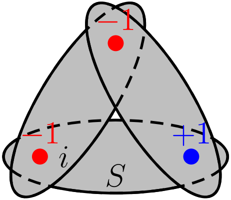

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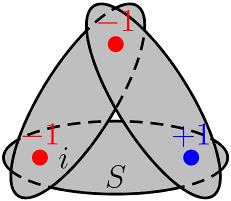

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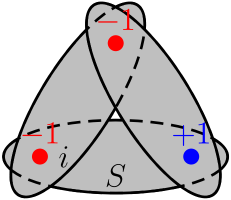

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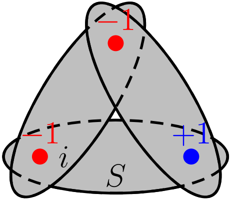

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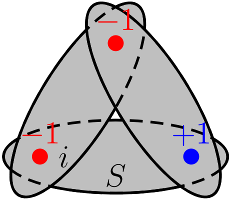

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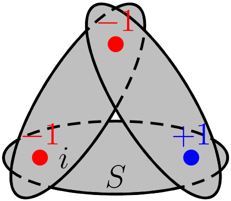

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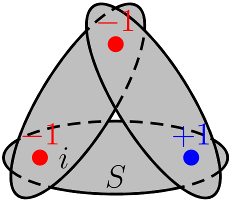

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Theorem (H., Rothvoss '18)

Suppose $n \geq \tilde{\Theta}(m^2)$, entries of $A \in \{0,1\}^{m \times n}$ chosen indep. with prob. p. Then with high probability disc$(A) \leq 1$.

Fourier Analysis

- Suppose $X \in \mathbb{Z}^m$ a random variable, $\theta \in \mathbb{R}^m$.
- We define

$$\hat{X}(\theta) = \mathbb{E}[e^{2\pi i \langle X, \theta \rangle}]$$

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- Fourier inversion formula: For $\lambda \in \mathbb{Z}^m$ we have

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Fourier Analysis for Discrepancy Theory

For fixed $A \in \{0,1\}^{m \times n}$:

- Choose $x \sim \{-1,1\}^n$ uniformly, and let $D = Ax$.

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- $X = D + R \in \mathbb{Z}^m$. We have

$$\begin{aligned}\Pr[X = 0] &= \int_{\theta \in [-\frac{1}{2}, \frac{1}{2})^m} \hat{X}(\theta) d\theta \\ &= \int_{\theta \in [-\frac{1}{2}, \frac{1}{2})^m} \hat{D}(\theta) \hat{R}(\theta) d\theta\end{aligned}$$

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where $A^j \in \{0,1\}^m$ the jth column of A.

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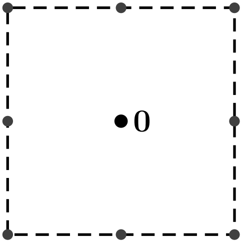

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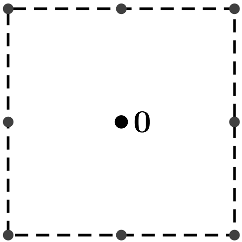

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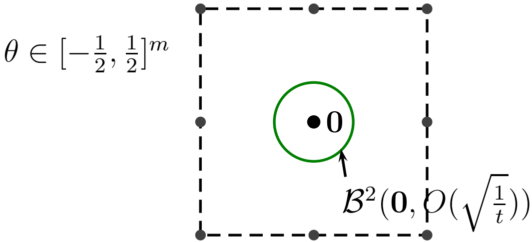

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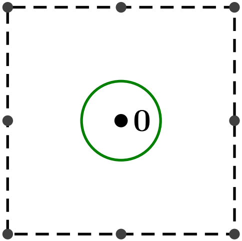

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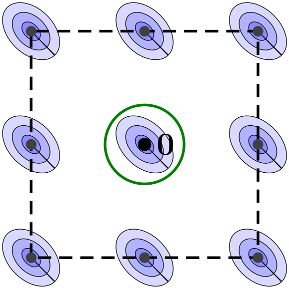

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- If $\langle A^j, \theta \rangle$ far from half-integral for many j, then $|\hat{D}(\theta)|$ small

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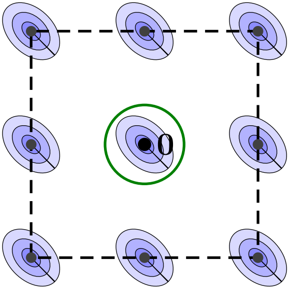

Analyzing $\hat{D}(\theta)$

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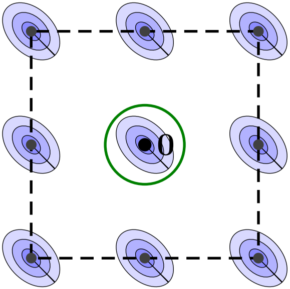

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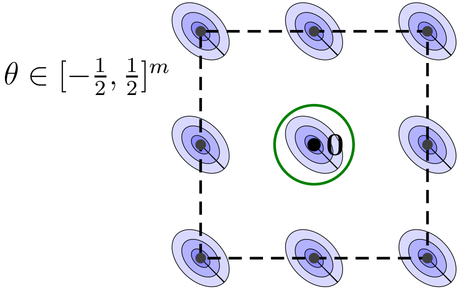

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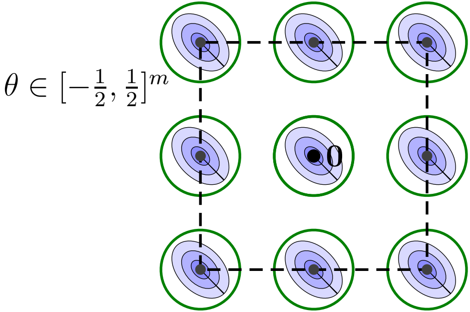

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Defining R

Given A, define $R \in \mathbb{Z}^m$ by

$$R_i = \begin{cases} 0 & \|A_i\|_1 \text{ even} \\ \pm 1 & \|A_i\|_1 \text{ odd (chosen uniformly)} \end{cases}$$

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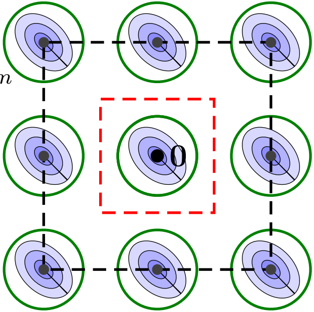

Analyzing $\hat{R}(\theta)$

We can compute

$$\hat{R}(\theta) = \mathbb{E}[e^{2\pi i \langle R, \theta \rangle}] = \prod_{\|A_i\|_1 \text{ odd}} \mathbb{E}[e^{2\pi i R_i \theta_i}] = \prod_{\|A_i\|_1 \text{ odd}} \cos(2\pi \theta_i)$$

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Summary

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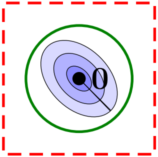

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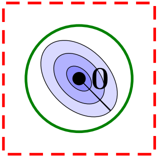

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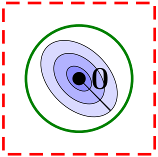

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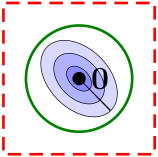

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Lemma

For $c > 0$ *some constant,* $\|\theta\|_\infty \le \frac{1}{4}$,

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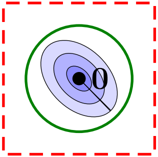
- But $\Phi_n = \int_{\|\theta\|_2 \ge \frac{1}{16\sqrt{t}}} |\hat{D}(\theta)| d\theta$ is exactly the quantity we wanted to bound.

Summary

$$\theta \in [-\tfrac{1}{4}, \tfrac{1}{4}]^m$$



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- Candidate R: For $i = 1, ..., m$ R_i chosen as a sum of Δ ind. $\{-1, 0, 1\}$ with $\Pr[R_i = 1] = \Pr[R_i = -1] = \frac{1}{4}$.

Thanks!