Finding Permutations with Prefix Targets

Alantha Newman Université Grenoble Alpes

Heiko Röglin Universität Bonn

Johanna Seif ENS Lyon

The Problem

Input:

- An ordered set of nonnegative integers $X = \{x_1, x_2, \ldots, x_n\}$ interspersed with free slots.
- Another set of nonnegative integers $Y = \{y_1 \geq y_2 \geq \cdots \geq y_n\}$.

Goal: Assign each element of Y to a free slot to minimize maximum value of a prefix.

The value of a prefix P is $X \cap P - Y \cap P$.

If every prefix has value in range $[\alpha, \beta]$, then value of solution is $\beta - \alpha$.

Example: $X = \{8, \_, 5, \_, 2, \_, 3, \_\}$, $Y = \{7, 6, 2, 3\}$.

Solution: $\{8, 7, 5, 6, 2, 2, 3, 3\}$ has value 8.

The Problem

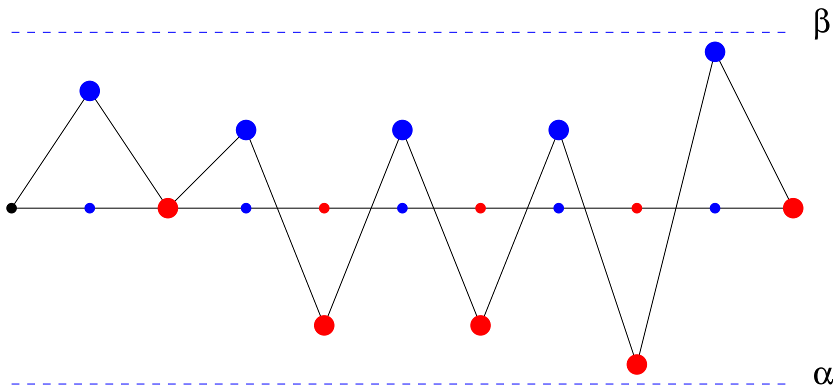
- We can assume the free slots alternate with the X values.
- If two x_i's are adjacent, add them together.
- If two free slots are adjacent, put a 0 between them.
- Generalizing 3-Partition:
  - X-input: $1, \_, 0, \_, 0, \_, 1, \_, 0, \_, 0, \_, 1, \ldots$.
  - Y-input: Instance of 3-Partition,
    i.e. $3n$ numbers strictly between $\frac{1}{4}$ and $\frac{1}{2}$ that sum to n.
  - Does every prefix have value in range $[0, 1]$?
- Hybrid Bin-Packing/Bin-Covering.

Problem: Geometric Interpretation

Input: $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1 \geq y_2 \geq \cdots \geq y_n\}$.

Goal: Find permutation π of Y to minimize distance between highest and lowest point, i.e. min $\beta - \alpha$.

$$\forall k: \quad \sum_{j-1}^{k} x_j - \sum_{j=1}^{k-1} \sum_{i=1}^{n} y_{\pi(i)} \ \leq \ \beta, \qquad \sum_{j-1}^{k} x_j - \sum_{j=1}^{k} \sum_{i=1}^{n} y_{\pi(i)} \ \geq \ \alpha.$$



Lower Bounds

Input: $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1 \geq y_2 \geq \cdots \geq y_n\}$.

$\mu_x = \text{max}_i\{x_i\}$, $\mu_y = \text{max}_i\{y_i\}$, $\mu = \text{max}\{\mu_x, \mu_y\}$.

$OPT \geq \mu$. But optimal solution may be much greater than μ.

Example: $OPT = n/2$, $\mu = 2$.

$$\begin{aligned} X &= \{\underbrace{2, 2, \ldots, 2}_{\frac{n}{2} \text{ entries}}, \underbrace{0, 0, \ldots, 0}_{\frac{n}{2} \text{ entries}}\}, \\ Y &= \{\underbrace{1, 1, \ldots, 1, 1, 1, \ldots, 1}_{n \text{ entries}}\}. \end{aligned}$$

Greedy/combinatorial algorithms can be terrible:

$$\begin{aligned} X &= \{\underbrace{2, 0, 2, 0, \ldots, 2, 0}_{\frac{n}{2} \text{ entries}}, \underbrace{2, \ldots, 2}_{\frac{n}{4} \text{ entries}}, \underbrace{0, \ldots, 0}_{\frac{n}{4} \text{ entries}}\}, \\ Y &= \{\underbrace{2, \ldots, 2}_{\frac{n}{4} \text{ entries}}, \underbrace{1, 1, 1, 1, \ldots, 1, 1}_{\frac{n}{2} \text{ entries}}, \underbrace{0, \ldots, 0}_{\frac{n}{4} \text{ entries}}\}. \end{aligned}$$

LP Relaxation

$$
\begin{aligned}
&\min \quad \beta - \alpha \\
\forall i \in [1,n] : \sum_{i=1}^{n} z_{ij} &= 1, \\
\forall i \in [1,n] : \sum_{j=1}^{n} z_{ij} &= 1, \\
\forall k : \sum_{j-1}^{k} x_j - \sum_{j=1}^{k-1} \sum_{i=1}^{n} y_i \cdot z_{ij} \ \le \ \beta, \qquad \sum_{j-1}^{k} x_j - \sum_{j=1}^{k} \sum_{i=1}^{n} y_i \cdot z_{ij} &\ge \alpha, \\
z_{ij} &\ge 0.
\end{aligned}
$$

Integer solution is a permutation matrix.
LP solution Z is doubly stochastic matrix.

$$
Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix}, \qquad z_j = \sum_{i=1}^{n} z_{ij} \cdot y_i.
$$

This Talk: Two Algorithms

Algorithm I.

- Uses iterative rounding (à la Karmarker-Karp).
- Has approximation factor $OPT_{LP} + \log n \cdot O(\mu_x + \mu_y)$.

Algorithm II.

- Transforms the support of Z
  (the doubly stochastic LP solution matrix).
- Has approximation factor $OPT_{LP} + \mu_y$.

Algorithm I: Rounding Up Items

Let OPT_{LP} denote optimal LP value for original instance.

Round up items (groups of eight consecutive items).

Substitute rounded-up values into LP solution to obtain
$OPT'_{LP} \leq OPT_{LP} + 8\mu_y$.

Let OPT_f be any feasible LP solution for rounded up instance such that:
$OPT_f \leq OPT'_{LP}$.

Replace rounded up values with original values into LP solution to obtain
$OPT'_f \leq OPT_f + 8\mu_y$.

Finally: $OPT'_f \leq OPT_{LP} + 16\mu_y$.

Algorithm I: Rearranging Items

Let OPT_{LP} denote optimal LP value for original instance.

Rearrange from alternating X-Y to alternating X^8-Y^8.

Obtain LP solution by rearranging columns:
$OPT'_{LP} \leq OPT_{LP} + 8\mu_x + 8\mu_y$.

Let OPT_f be any feasible LP solution for rounded up instance such that:
$OPT_f \leq OPT'_{LP}$.

Put columns of the LP solution back into alternating order:
$OPT'_f \leq OPT_f + 8\mu_y$.

Finally: $OPT'_f \leq OPT_{LP} + 16\mu_y + 8\mu_x$.

Algorithm I: Iterative Rounding

Round up items.

Rearrange items.

Obtain permutation instance on $n/8$ variables with new LP.
Let z'_{ij} denote new variables and y'_i denote rounded-up items and rearranged items.

Assign y'_i's to positions in the 8×8 boxes such that resulting matrix is doubly stochastic and so that integer parts are unit.

Un-rearrange and round down.

Total cost: $OPT_{LP} + \log n \cdot O(\mu_x + \mu_y)$.

New LP Relaxation

$$
\begin{aligned}
& \min \quad \beta - \alpha \\
\forall i \in [1, n'] : \sum_{i=1}^{n} z_{ij} &= 8, \\
\forall i \in [1, n'] : \sum_{j=1}^{n} z_{ij} &= 8, \\
\forall k : \sum_{j-1}^{k} x_j - \sum_{j=1}^{k-1} \sum_{i=1}^{n'} y'_i \cdot z_{ij} \le \beta, \quad \sum_{j-1}^{k} x_j - \sum_{j=1}^{k} \sum_{i=1}^{n'} y'_i \cdot z_{ij} &\ge \alpha, \\
z_{ij} &\ge 0.
\end{aligned}
$$

There are $4n'$ constraints $\Rightarrow$ at most $4n'$ nonzero variables in a BFS.
If $n' = \frac{n}{8} \Rightarrow$ there are $\frac{n}{2}$ nonzero variables in a BFS.
But total weight is n.
So at least half the weight belongs to the integer parts.

LP Relaxation

$$
\begin{aligned}
&& \min \quad & \beta - \alpha \\
&& \forall i \in [1,n] : \sum_{i=1}^{n} z_{ij} \quad &= \quad 1, \\
&& \forall i \in [1,n] : \sum_{j=1}^{n} z_{ij} \quad &= \quad 1, \\
\forall k : \sum_{j-1}^{k} x_j - \sum_{j=1}^{k-1} \sum_{i=1}^{n} y_i \cdot z_{ij} \ \leq \ \beta, && \sum_{j-1}^{k} x_j - \sum_{j=1}^{k} \sum_{i=1}^{n} y_i \cdot z_{ij} \quad &\geq \quad \alpha, \\
&& z_{ij} \quad &\geq \quad 0.
\end{aligned}
$$

Integer solution is a permutation matrix.
LP solution Z is doubly stochastic matrix.

$$
Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{nn} \end{pmatrix}, \qquad z_j = \sum_{i=1}^{n} z_{ij} \cdot y_i.
$$

Algorithm II: Approximating z_j's

Recall LP relaxation.

$$z_j = \sum_{i=1}^{n} z_{ij} \cdot y_i.$$

Goal: Find permutation of $Y = \{y'_1, y'_2, \ldots y'_n\}$ such that:

$$\sum_{j \in [1,\ell]} z_j \leq \sum_{j \in [1,\ell]} y'_j \leq \sum_{j \in [1,\ell]} z_j + \gamma.$$

Solution size at most $\Rightarrow \beta - \alpha + \gamma \ = \ OPT + \gamma$.

Relaxation

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{nn} \end{pmatrix}, \qquad z_j = \sum_{i=1}^{n} z_{ij} \cdot y_i.$$

Approach:

Decompose Z into permutation matrices à la Birkoff-von Neumann.

Bad example: $Y = \{\underbrace{1, 1, \dots, 1}_{n-k}, \underbrace{B, \dots, B}_{k}\}$. $x_i = \frac{(k \cdot B) + (n-k)}{n} = z_j$.

$$\{B, B, \dots, B, 1, 1, \dots 1, 1, 1\}$$
$$\{1, B, B, \dots, B, 1, 1, \dots 1, 1\}$$
$$\{1, 1, B, B, \dots, B, 1, 1, \dots 1\}.$$

$$\{1, 1, \dots 1, B, 1 \dots, 1, B, 1, \dots 1\}.$$

Rounding: Transformation

We transform doubly stochastic Z into a doubly stochastic matrix T.

Preserves the weighted column sums $\{z_j\}$. (Still feasible for LP.)

We say a row i is *finished* at column j, if columns 1 through j sum to 1 in row i.

Matrix T will have **consecutiveness property**: In each column, there are at most two non-zero entries that are not in *finished rows*, and in between them, all the rows are finished.

$$T = \begin{pmatrix} 0 & .7 & 0 & .3 & 0 & \ldots \\ .5 & .3 & .2 & 0 & \ldots & \ldots \\ .5 & 0 & .5 & 0 & \ldots & \ldots \\ 0 & 0 & .3 & .7 & 0 & \ldots \\ 0 & 0 & 0 & 0 & .2 & \ldots \\ 0 & 0 & 0 & 0 & .8 & \ldots \end{pmatrix}$$

Rounding: Transformation

If $z_a, z_b, z_c > 0$ all appear in an unfinished row. Then:

increase $z_b \to z_b + \delta$

decrease $z_a \to z_a - \delta \cdot \frac{y_b - y_c}{y_a - y_c}$

decrease $z_c \to z_c - \delta \cdot \frac{y_a - y_b}{y_a - y_c}$

For some column (to the right) where row of b has value at least δ:

decrease value in row b by δ

increase values in rows a and c

Rounding: Transformation Example

$$Z = \begin{pmatrix} 0 & .6 & .4 & 0 & 0 & 0 \\ .3 & 0 & .1 & .4 & .2 & 0 \\ .4 & .4 & .2 & 0 & 0 & 0 \\ .3 & 0 & 0 & .6 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}, Y = \begin{pmatrix} 16 \\ 8 \\ 6 \\ 4 \\ 2 \\ 1 \end{pmatrix},$$

$$Z^t Y = \begin{pmatrix} 6 & 12 & 9 & 5.6 & 3 & 1.4 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & .6 & .4 & 0 & 0 & 0 \\ .1 & .2 & .1 & .4 & .2 & 0 \\ .8 & 0 & .2 & 0 & 0 & 0 \\ .1 & .2 & 0 & .6 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}, \begin{pmatrix} 0 & .6 & .4 & 0 & 0 & 0 \\ 0 & .2 & .2 & .4 & .2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .1 & .6 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}, \begin{pmatrix} 0 & .5 & .4 & .1 & 0 & 0 \\ 0 & .5 & .2 & .1 & .2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & .8 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}$$

Rounding: Transformation Example

$$Z = \begin{pmatrix} 0 & .6 & .4 & 0 & 0 & 0 \\ .3 & 0 & .1 & .4 & .2 & 0 \\ .4 & .4 & .2 & 0 & 0 & 0 \\ .3 & 0 & 0 & .6 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}, Y = \begin{pmatrix} 16 \\ 8 \\ 6 \\ 4 \\ 2 \\ 1 \end{pmatrix},$$

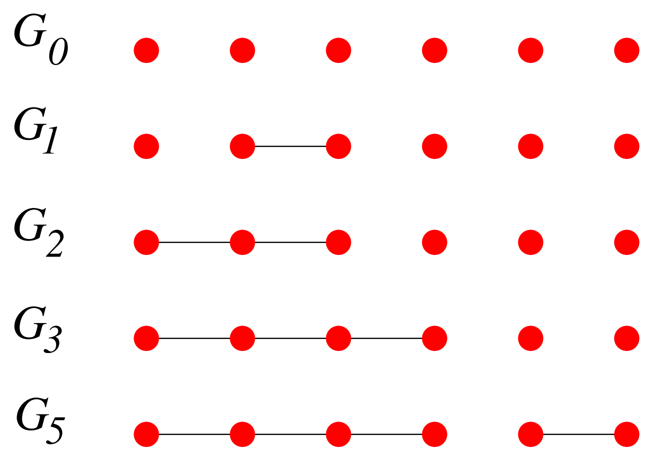
$$Z^t Y = \begin{pmatrix} 6 & 12 & 9 & 5.6 & 3 & 1.4 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & .6 & .4 & 0 & 0 & 0 \\ .1 & .2 & .1 & .4 & .2 & 0 \\ .8 & 0 & .2 & 0 & 0 & 0 \\ .1 & .2 & 0 & .6 & .1 & 0 \\ 0 & 0 & .3 & 0 & .3 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & .5 & .25 & .15 & .1 & 0 \\ 0 & .5 & .5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .25 & .75 & 0 & 0 \\ 0 & 0 & 0 & .1 & .5 & .4 \\ 0 & 0 & 0 & 0 & .4 & .6 \end{pmatrix}$$

Rounding: Using T

$$T = \begin{pmatrix} 0 & .7 & 0 & .3 & 0 & \dots \\ .5 & .3 & .2 & 0 & \dots & \dots \\ .5 & 0 & .5 & 0 & \dots & \dots \\ 0 & 0 & .3 & .7 & 0 & \dots \\ 0 & 0 & 0 & 0 & .2 & \dots \\ 0 & 0 & 0 & 0 & .8 & \dots \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

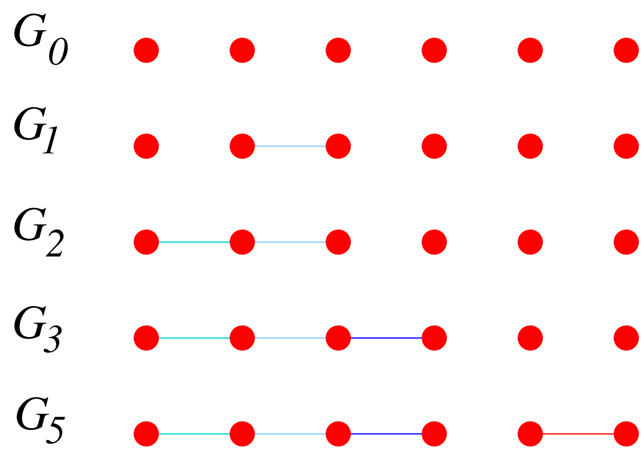
For each column, connect the vertices (e.g. add a clique on vertices) in rows of support.



Rounding: Using T

$$T = \begin{pmatrix} 0 & .7 & 0 & .3 & 0 & \dots \\ .5 & .3 & .2 & 0 & \dots & \dots \\ .5 & 0 & .5 & 0 & \dots & \dots \\ 0 & 0 & .3 & .7 & 0 & \dots \\ 0 & 0 & 0 & 0 & .2 & \dots \\ 0 & 0 & 0 & 0 & .8 & \dots \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

For each column, connect the vertices (e.g. add a clique on vertices) in rows of support.



Rounding

Main Theorem: Rounding results in permutation of Y with guarantee $OPT + \mu$.

Main idea: For each prefix $[1, \ell]$, we want to approximate $\sum_{j \in [1,\ell]} z_j$.
Finished block contribute equally to T and R.

$$T = \begin{pmatrix} 0 & .7 & 0 \\ .5 & .3 & .2 \\ .5 & 0 & .5 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In prefix $[1, 3]$, the difference is $.3(y_1 - y_4) \leq (y_1 - y_4)$.

Each unfinished block B contributes at most $(y_b - y_a)$ to R than to T.
y_b (y_a) is max (min) y-value in B.

Total difference between T and R over all blocks is:

$$\sum_{\text{all } B} (y_b - y_a) = \mu_y.$$

Open Problems:
Better Upper and Lower Bounds in Higher Dimensions

Input: $\{v_1, v_2, \ldots, v_n\}, v_i \in \mathbb{R}^d, ||v_i|| \leq 1$.

Steinitz Problems: Input + fact that $\sum v_i = 0$.

Find permutation of vectors so that each prefix is small.

Signed Series Problems: (Upper bound on Steinitz constant)

Assign signs to each vector so that all prefixes are small.

2-dimensional problem: 2 [Lund, Magazinov 2015]

Alternating Problems: Input + $\sum v_i = 0$, hyperplane H.

Find permutation of vectors alternating between sides of hyperplane.

Gasoline Problems: Fix half the vectors. Permute the other half.

Find permutation so each prefix is small.