Balancing Vectors in Any Norm

Aleksandar (Sasho) Nikolov

University of Toronto

Based on joint work with

Daniel Dadush, Kunal Talwar, and Nicole Tomczak-Jaegermann

Outline

Discrepancy

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathrm{disc}(U, \|\cdot\|_\infty) = \min_{\varepsilon \in \{\pm 1\}^N} \|U\varepsilon\|_\infty$$

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Natural to consider arbitrary norms: any norm can be written as $\|U \cdot\|_\infty$.

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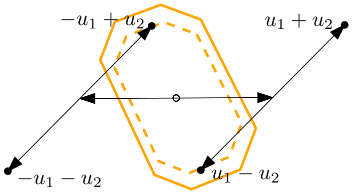
Most combinatorial discrepancy bounds are implied by geometric vector balancing arguments.

The Vector Balancing Problem

Given $u_1, \ldots, u_N \in \mathbb{R}^n$, and symmetric convex body $K \subset \mathbb{R}^n$ ($K = -K$), find the smallest t such that

$$\exists\ \varepsilon_1, \ldots, \varepsilon_N \in \{-1, +1\} : \varepsilon_1 u_1 + \ldots + \varepsilon_N u_N \in tK$$



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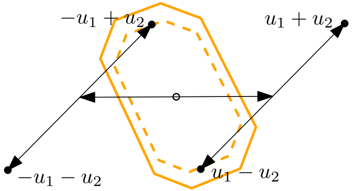

Minkowski Norm: $\|x\|_K = \inf\{t : x \in tK\}$; $t = \mathrm{disc}((u_i)_{i=1}^N, \|\cdot\|_K)$.

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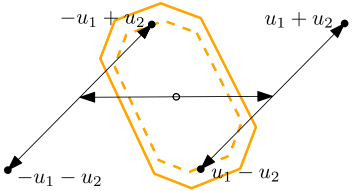

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Vector Balancing Constant: worst case over sequences in C

$$\mathrm{vb}(C, K) = \sup\left\{ \mathrm{disc}(U, \|\cdot\|_K) : N \in \mathbb{N}, u_1, \ldots, u_N \in C, U = (u_i)_{i=1}^N \right\}$$

Questions and Prior Results

- [Dvoretzky, 1963] “What can be said” about $\mathrm{vb}(K, K)$?
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- [Banaszczyk, 1998] $vb(B_2^n, K) \leq 5$ if K has Gaussian measure $\gamma_n(K) \geq \frac{1}{2}$
- *Komlós Problem*: Prove or disprove $\mathrm{vb}(B_2^n, B_\infty^n) \lesssim 1$.
    - Banaszczyk's theorem implies $\mathrm{vb}(B_2^n, B_\infty^n) \lesssim \sqrt{\log 2n}$.

Vector Balancing and Rounding

For any $w \in [0,1]^N$, any $U = (u_i)_{i=1}^N$, $u_i \in C$, and any symmetric convex K, there exists a $x \in \{0,1\}^N$ such that

$$\|Ux - Uw\|_K \leq \mathrm{vb}(C,K).$$

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  - The proof implies an efficient algorithm to compute $\varepsilon \in \{-1, 1\}^N$ given $u_1, \ldots, u_N \in C$, so that $\|\varepsilon_1 u_1 + \ldots + \varepsilon_N u_N\|_K \lesssim (1 + \log n)\,\mathrm{vb}(C, K)$.
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Prior work [Bansal, 2010; Nikolov and Talwar, 2015] implies bounds which deteriorate with the number of facets of K.

Outline

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Issue: $\text{disc}(U, K) = \text{disc}(U, \|\cdot\|_K)$ is

- not robust to slight changes in U (e.g. repeat each column)
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Hereditary discrepancy is a robust analog of discrepancy:

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Observation:

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The Volume Lower Bound

Define $L = \{x \in \mathbb{R}^N : Ux \in K\}$: the set of "good x".

- $\mathrm{disc}(U, K) = \min\{t : tL \cap \{-1, 1\}^N \neq \emptyset\}$.

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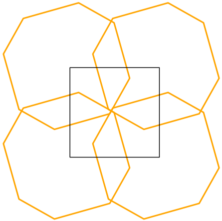
[Lovász, Spencer, and Vesztergombi, 1986]:
If $t = \text{hd}(U, K)$, then $[0, 1]^N \subseteq \bigcup_{x \in \{0,1\}^N} (x + tL)$.



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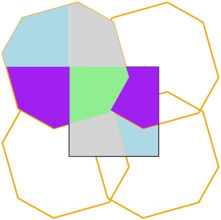

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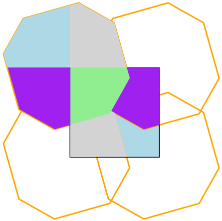

[Banaszczyk, 1993]:

$$1 = \mathrm{vol}([0, 1]^N) \geq \mathrm{vol}(tL) = t^N \mathrm{vol}(L) \iff \mathrm{hd}(U, K) \geq \mathrm{vol}(L)^{-1/N}.$$

A Hereditary Volume Lower Bound

A simple strengthening:

$$\mathrm{hd}(U,K) \geq \mathrm{volLB}(U,K) = \max_{S \subseteq [N]} \mathrm{vol}(\{x \in \mathbb{R}^S : U_S x \in K\})^{-1/|S|}.$$

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Theorem

For any $n \times N$ matrix U, and any symmetric convex $C, K \subset \mathbb{R}^n$,

$$\mathrm{volLB}(U, K) \leq \mathrm{hd}(U, K) \lesssim (1 + \log n) \cdot \mathrm{volLB}(U, K)$$

$$\mathrm{volLB}(C, K) \leq \mathrm{vb}(C, K) \lesssim (1 + \log n) \cdot \mathrm{volLB}(C, K)$$

Rothvoß's Algorithm

Algorithm [Rothvoß, 2014]: given $K \subset \mathbb{R}^n$,

1. Sample a standard Gaussian $G \sim N(0, I_n)$;
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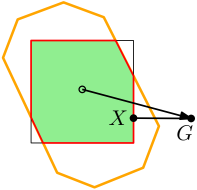
   $X = \arg\min\{\|x - G\|_2^2 : x \in K \cap [-1,1]^n\}$.



Goal: $|\{i : X_i \in \{-1,+1\}\}| \geq \alpha n$ for a constant α.
(X is a *partial coloring*.)

Intuition: If K is "big enough," then in an average direction $\partial[-1,1]^n$ is closer to the origin than ∂K and is more likely to be hit by X.

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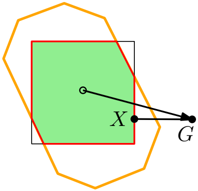

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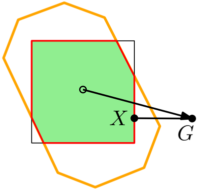

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Tightness of the Volume Lower Bound

Need to show: for any $U \in \mathbb{R}^{n \times N}$ and symmetric convex $K \subset \mathbb{R}^n$

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Main Challenge: Show that the conditions of Rothvoß's algorithm are satisfied.

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Theorem (Structural result)

For any δ there exists a $m = m(\delta)$ so that the following holds. Let L be a symmetric convex body s.t. $\mathrm{vol}(L \cap \mathbb{R}^S) \geq 1$ for all $S \subseteq [n]$. There exists a subspace W of dimension $(1-\delta)n$ for which

$$\gamma_W((mL) \cap W) \geq e^{-\delta n}.$$

Apply to $L = \mathrm{volLB}(I_n, K) \cdot K$ to get that the conditions of Rothvoß's algorithm are satisfied.

Proof Ideas

Generally applicable strategy:

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Theorem (Regular M-ellipsoid, [Milman, 1986; Pisier, 1989])

For any symmetric convex $L \subseteq \mathbb{R}^n$ there exists an ellipsoid E such that for any $t \geq 1$

$$\max\{N(L, tE), N(E, tL)\} \leq e^{cn/t},$$

where c is a constant.

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- $L \cap \mathbb{R}^S$ has large volume $\implies$ $E \cap \mathbb{R}^S$ has large volume.
- $E \cap W$ has large Gaussian measure $\implies$ $L \cap W$ has large Gaussian measure.

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- True for ellipsoids and reduces to the Restricted Invertibility Principle.
- True for general bodies K if we replace $\mathbb{R}^S$ with an arbitrary subspace W and $|S|$ with $\dim W$.

Outline

Upper Bounds from Banaszczyk's Theorem

We showed how to efficiently compute near optimal signs $\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}$ for any $u_1, \ldots, u_N$.

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- $\mathrm{vb}(B_2^n, K) \lesssim \mathbb{E}\|G\|_K$.
- $\mathrm{vb}(C, K) \lesssim (\mathbb{E}\|G\|_K) \cdot \mathrm{diam}_{\ell_2}(C)$.

Last bound can be very loose! Can we do better?

A Better Upper Bound

Idea: Map C into B_2^n using a linear map.

$$\lambda(C,K) = \inf\{(\mathbb{E}\|G\|_{T(K)}) \cdot \mathrm{diam}_{\ell_2}(T(C)) : T \text{ a linear map}\}.$$

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- Take a linear map T achieving $\lambda(C, K)$;
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- $\mathrm{vb}(C, K) = \mathrm{vb}(T(C), T(K))$ and apply Banaszczyk's theorem.

Tightness of the Upper Bound

Theorem

For any symmetric convex $C, K \subset \mathbb{R}^n$*,*

$$\frac{\lambda(C,K)}{(1+\log n)^{5/2}} \lesssim \mathrm{vb}(C,K) \lesssim \lambda(C,K).$$

Moreover, given membership oracle access to K *and a vertex representation of* C*, we can efficiently compute* $\lambda(C,K)$*.*

For a matrix $U \in \mathbb{R}^{n \times N}$, we can take $C = \mathrm{conv}\{\pm u_1, \dots, \pm u_N\}$, and then $\lambda(C,K)$ approximates $\mathrm{hd}(U,K)$.

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Proof outline:

1. Formulate $\lambda(C,K)$ as a convex minimization problem;
2. Derive the Lagrange dual: an equivalent maximization problem;
3. Relate dual solutions to the volume lower bound.

Convex Formulation

$\|x\|_{T(K)} = \|T^{-1}x\|_K$

First attempt: $\inf\{\mathbb{E}\|T^{-1}G\|_K : \mathrm{diam}_{\ell_2}(T(C)) \leq 1\}$

- *Not convex*: the objective is ∞ for $T = 0$ and finite for any invertible T, but $0 = \frac{1}{2}(T + (-T))$.

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Formulation:

$$\begin{aligned} \lambda(C,K) = &\inf f(A) \\ \text{s.t. } &\langle x, Ax\rangle \le 1 \quad \forall x \in C \\ &A \succ 0. \end{aligned}$$

- $f(A) = \mathbb{E}\|T^{-1}G\|_K$ for any T such that $T^*T = A$;
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- The first constraint encodes $\mathrm{diam}_{\ell_2}(T(C)) \le 1$: $\langle x, Ax\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|_2^2$.

Properties of the Formulation

- The function $f(A)$ is convex in A, and the constraints are also convex;
- **Lagrange Duality**: there exists an *equivalent* dual maximization problem, whose value also equals $\lambda(U, C)$;

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Computation: The convex optimization problem can be solved using the ellipsoid method, given a membership oracle for K and a vertex representation of C.

Outline

Conclusion

In this work:

- Tightness of natural upper and lower bounds for vector balancing.
- Efficient algorithms to find nearly optimal vector balancing signs, and to compute $\mathrm{vb}(C, K)$, and hereditary discrepancy with respect to any norm.
- Our results strongly use the geometry of the underlying discrepancy problem.

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- Tightness of natural upper and lower bounds for vector balancing.
- Efficient algorithms to find nearly optimal vector balancing signs, and to compute $\mathrm{vb}(C, K)$, and hereditary discrepancy with respect to any norm.
- Our results strongly use the geometry of the underlying discrepancy problem.

Open questions:

- Does $\mathrm{volLB}(C, K)$ give lower bounds on partial colorings?
- $\mathrm{vb}(K, K) \asymp \mathrm{volLB}(K, K)$? (True for ℓ_p.)
- Can the bounds for $\lambda(C, K)$ be improved?

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