Discrepancy Theory and Applications to Bin Packing

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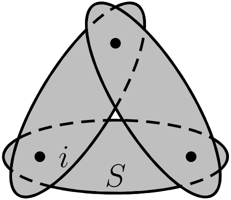
Joint work with Becca Hoberg

UNIVERSITY *of* WASHINGTON

Discrepancy theory



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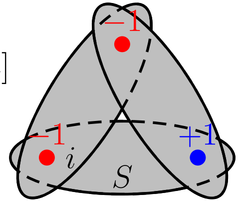

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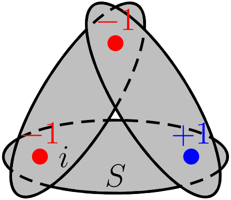

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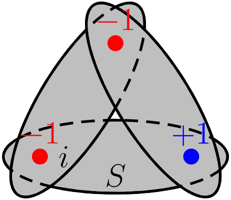

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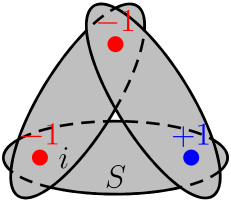

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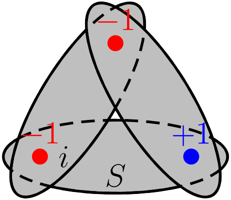

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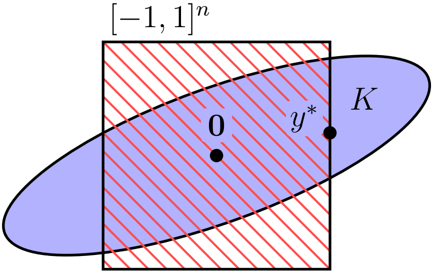
Main method: Iteratively find a **partial coloring**.

Discrepancy algorithm

Theorem (R., FOCS 2014)

For a **convex symmetric set** $K \subseteq \mathbb{R}^n$ with $\Pr[\text{gaussian} \in K] \geq e^{-\Theta(n)}$, one can find a $y \in K \cap [-1,1]^n$ with $|\{i : y_i = \pm 1\}| \geq \Theta(n)$ in **poly-time**.



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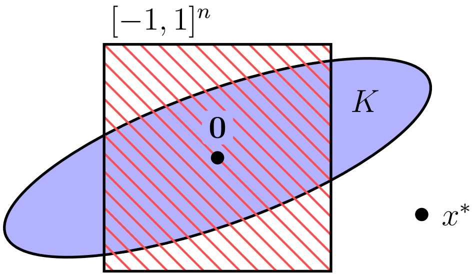

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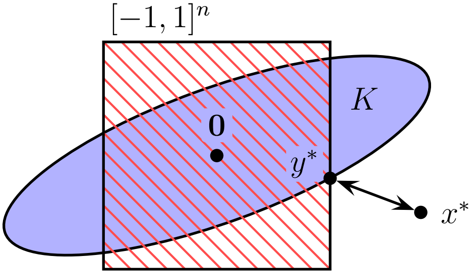
(2) compute $y^* = \text{argmin}\{\|x^* - y\|_2 \mid y \in K \cap [-1,1]^n\}$



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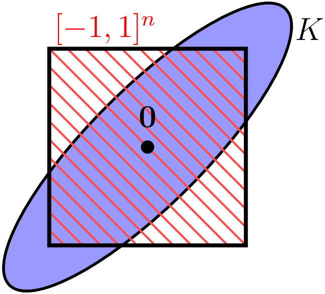

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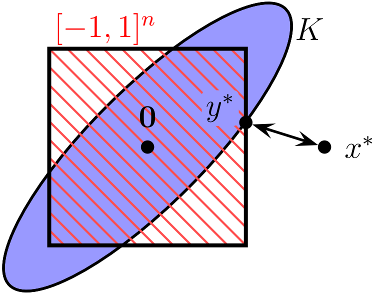
- W.h.p. $\|x^* - y^*\|_2 \geq \frac{1}{5}\sqrt{n}$



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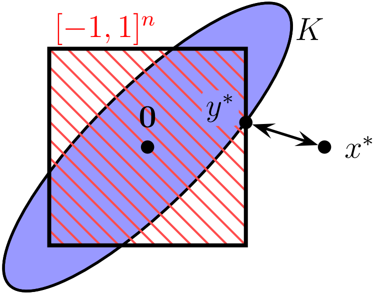

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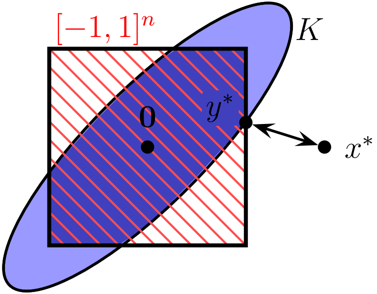

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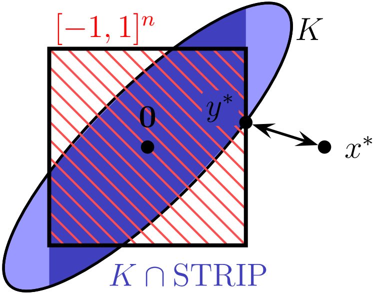

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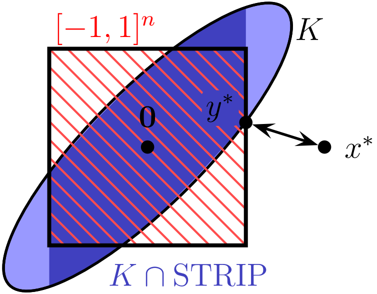

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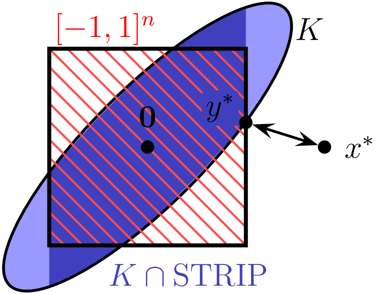
- Then $\mathbb{E}[\text{dist}(\text{gaussian}, K \cap \text{STRIP})] \leq o(\sqrt{n})$. Contradiction!

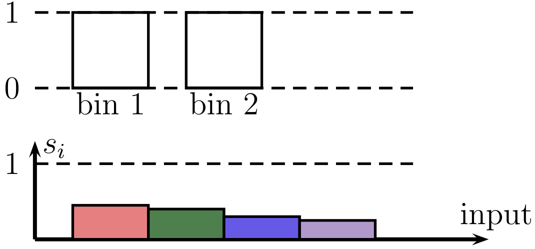


Application to Bin Packing

Bin Packing

Input: Items with sizes $s_1, \dots, s_n \in [0, 1]$

Goal: Pack items into minimum number of **bins** of size 1.



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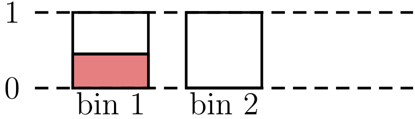

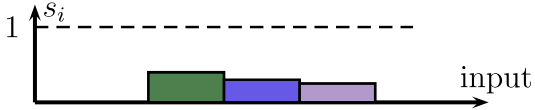

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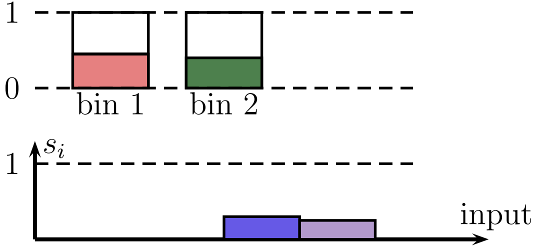

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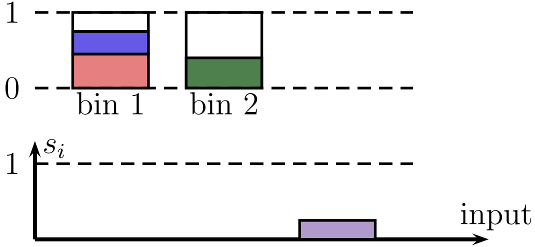

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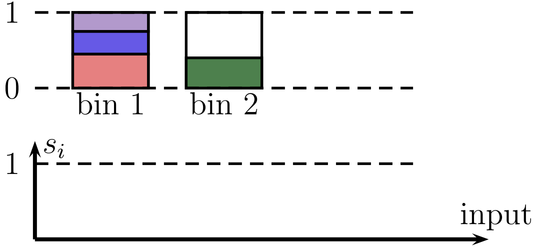

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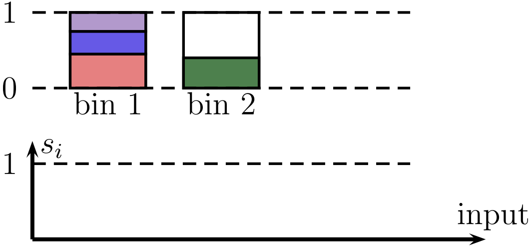

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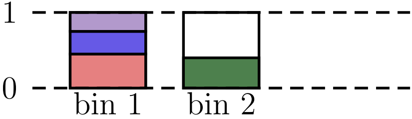

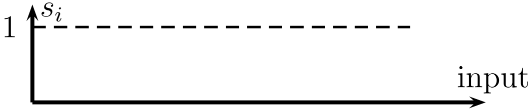

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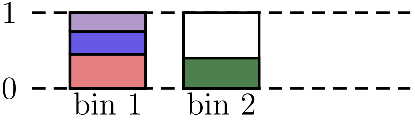

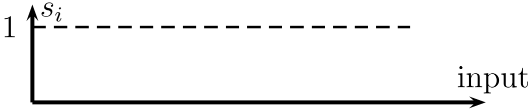

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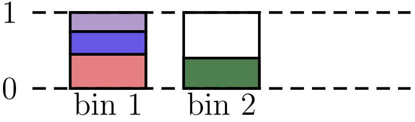

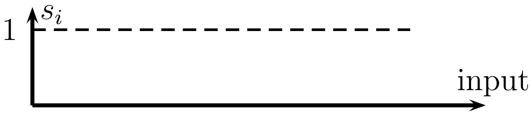

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- [Karmarkar & Karp '82]: $APX \leq OPT + O(\log^2 OPT)$ in poly-time

The Gilmore Gomory LP relaxation

- $b_i = \#\text{items with size } s_i$
- **Feasible patterns**:

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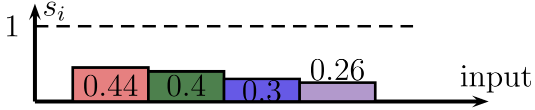
- Can find x with $\mathbf{1}^T x \leq OPT_f + \delta$ in time $\text{poly}(\|b\|_1, \frac{1}{\delta})$

The Gilmore Gomory LP - Example



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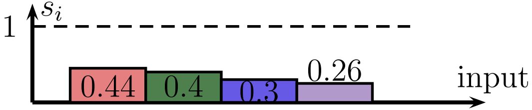

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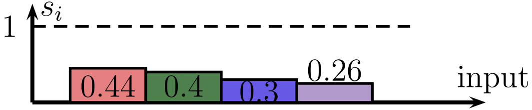

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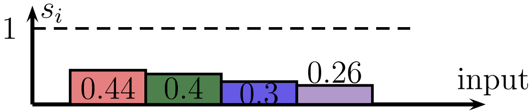

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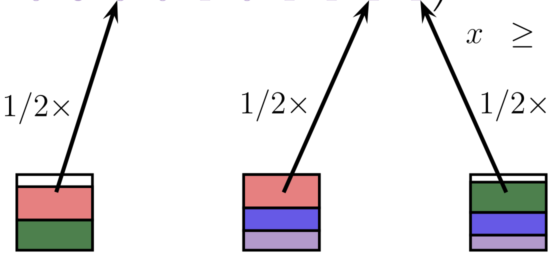
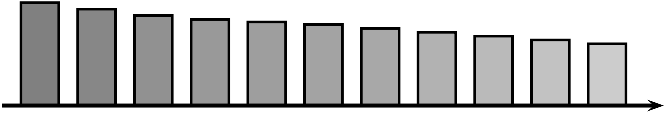


Karmarkar-Karp's Grouping

input:



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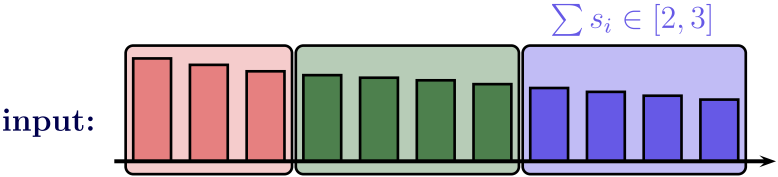

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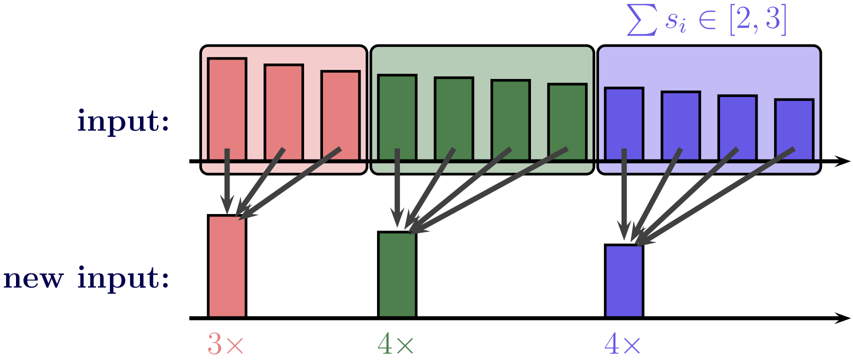

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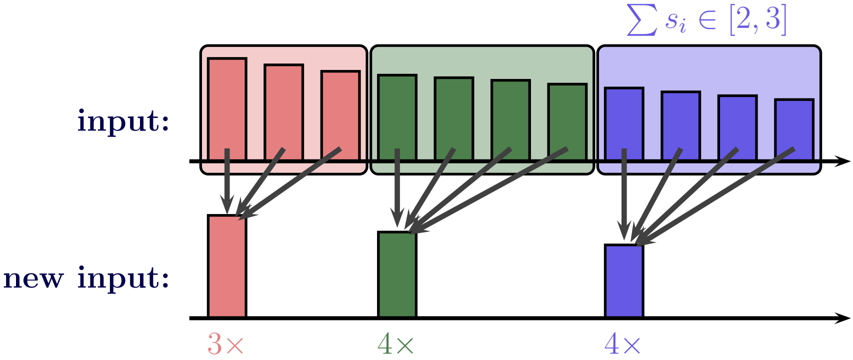

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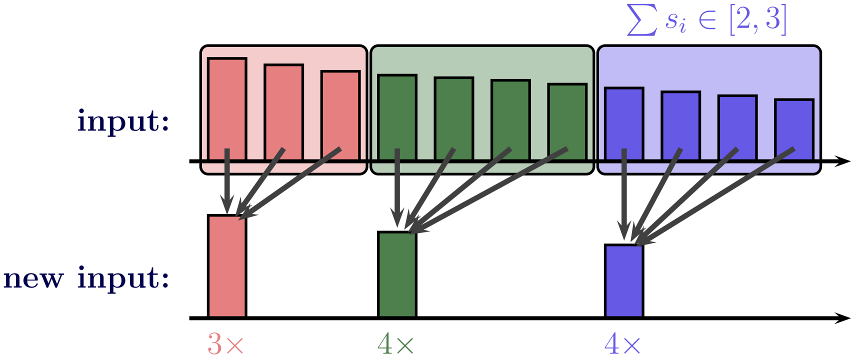

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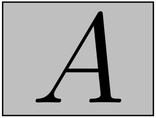

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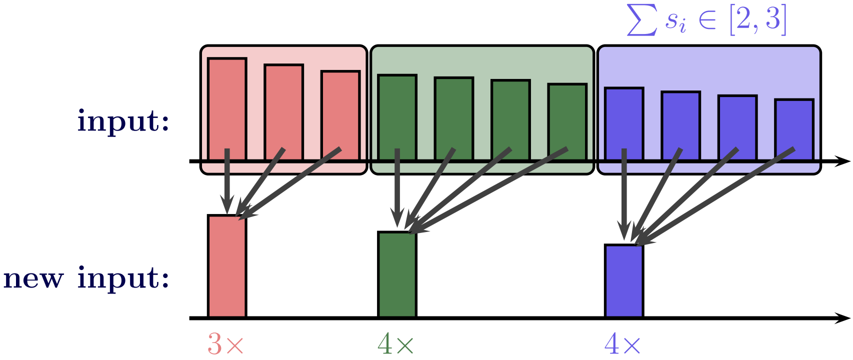
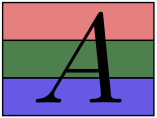
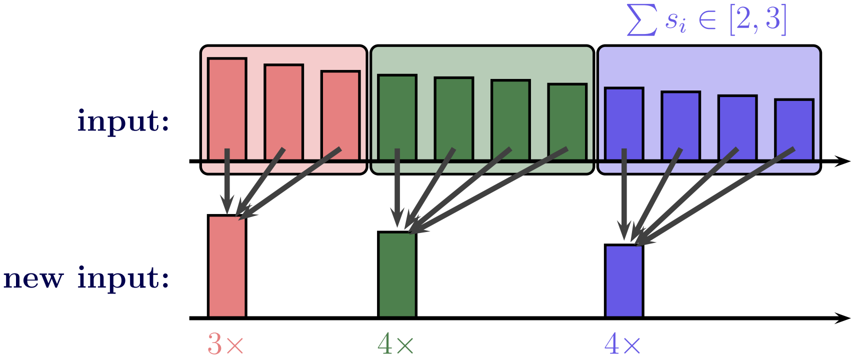


- increases OPT by $O(\log n)$



Karmarkar-Karp's Grouping



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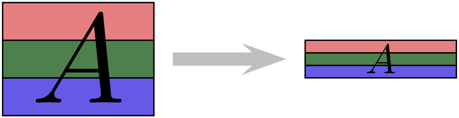
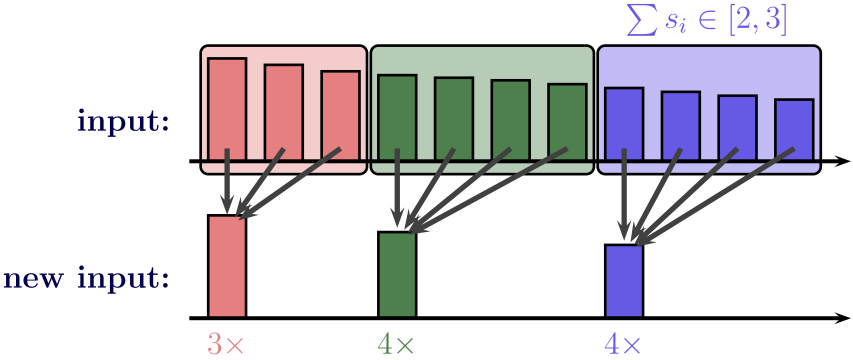
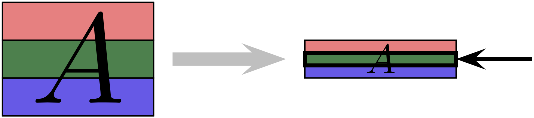


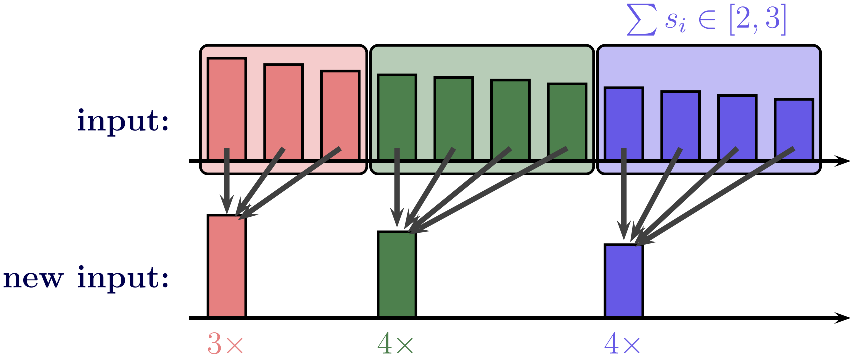
Karmarkar-Karp's Grouping



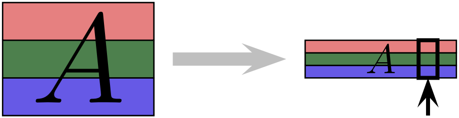
- increases OPT by $O(\log n)$
- row sum $\cdot\, s_i \geq 2$



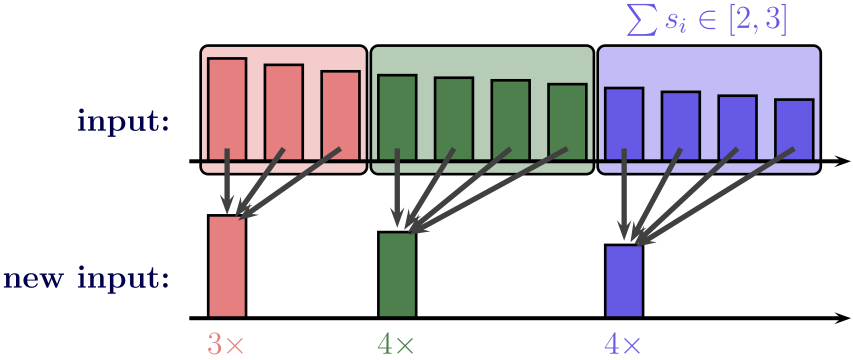
Karmarkar-Karp's Grouping



- increases OPT by $O(\log n)$
- row sum $\cdot\, s_i \geq 2 \quad \longleftrightarrow \quad$ column sum (w.r.t s_i) ≤ 1



Karmarkar-Karp's Grouping



- increases OPT by $O(\log n)$
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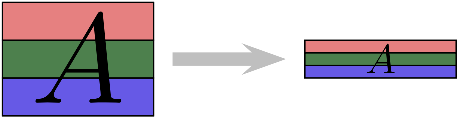
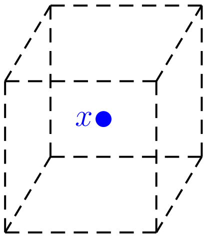
- # constraints $\leq \frac{1}{2}$support(x)

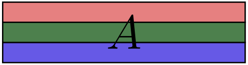


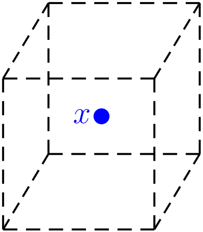
Karmarkar-Karp algo (2)



Karmarkar-Karp algo (2)

- After grouping: # constraints $\leq \frac{1}{2}|\text{supp}(x)|$





Karmarkar-Karp algo (2)

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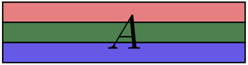
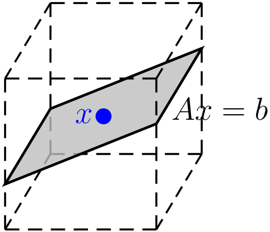
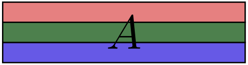
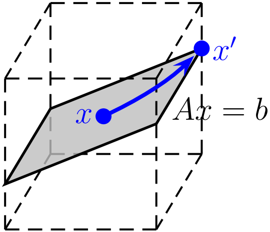




Karmarkar-Karp algo (2)

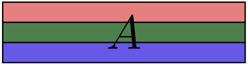
- After grouping: # constraints $\leq \frac{1}{2}|\text{supp}(x)|$
- Move to **basic solution** x' with $Ax' = Ax$

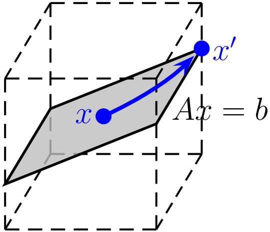




Karmarkar-Karp algo (2)

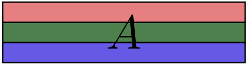
- After grouping: # constraints $\leq \frac{1}{2}|\mathrm{supp}(x)|$
- Move to **basic solution** x' with $Ax' = Ax$
- $|\mathrm{supp}(x' - \lfloor x' \rfloor)| \leq \frac{1}{2}|\mathrm{supp}(x)|$

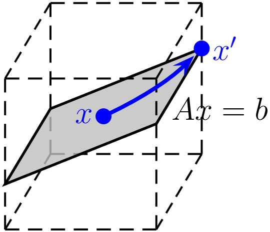




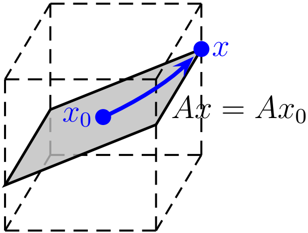
Karmarkar-Karp algo (2)

- After grouping: # constraints $\leq \frac{1}{2}|\text{supp}(x)|$
- Move to **basic solution** x' with $Ax' = Ax$
- $|\text{supp}(x' - \lfloor x' \rfloor)| \leq \frac{1}{2}|\text{supp}(x)|$
- Repeat $O(\log n)$ times $\rightarrow \boldsymbol{O(\log^2 n)}$

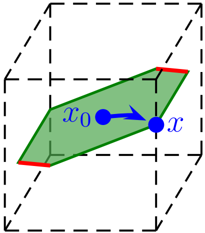




Applying Discrepancy to Bin Packing



Karmarkar & Karp



Hoberg & R.

Theorem (R. FOCS '13, Hoberg-R. SODA '17)

One can find a packing with $\boldsymbol{OPT + O(\log n)}$ bins in **poly-time**.

Beating Karmarkar & Karp

Assumptions:

- Fractional solution $x \in [0,1]^n$, matrix A is $O(n) \times n$

Beating Karmarkar & Karp

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Beating Karmarkar & Karp

Assumptions:

- Fractional solution $x \in [0,1]^n$, matrix A is $O(n) \times n$
- Item sizes in $[\frac{1}{k}, \frac{2}{k}]$
- OPT_f increases whenever glue 2 items

Beating Karmarkar & Karp

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Goal: Replace x with y so that $|\text{frac}(y)| \leq \frac{n}{2}$.

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Karmarkar-Karp: Loose $O(k)$ items $= O(1)$ bins

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Goal: Replace x with y so that $|\text{frac}(y)| \leq \frac{n}{2}$.
Karmarkar-Karp: Loose $O(k)$ items $= O(1)$ bins
We: Loose $O(k^{15/16})$ items $= O(k^{-1/16})$ bins

Beating Karmarkar & Karp

Assumptions:

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Karmarkar-Karp: Loose $O(k)$ items $= O(1)$ bins
We: Loose $O(k^{15/16})$ items $= O(k^{-1/16})$ bins

Lemma (Lovett-Meka)

For $x \in [0,1]^n$, vectors v_i, parameters $\lambda_i \geq 0$ with

$$\sum_{i=1}^{m} \exp(-\lambda_i^2/16) \leq \frac{n}{16}$$

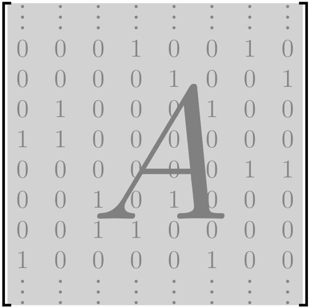
can find partial coloring $y \in [0,1]^n$ with at least half the entries in $\{0,1\}$ and $|\langle v_i, x - y\rangle| \leq \lambda_i \|v_i\|_2$.

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$A$$

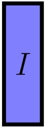
Applying the Partial Coloring Lemma



- **Given:** x. **Find:** y with $|(\sum_{j \leq i} A_j)(x-y)|$ small

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

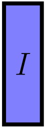


- **Given:** x. **Find:** y with $|(\sum_{j \leq i} A_j)(x-y)|$ small

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

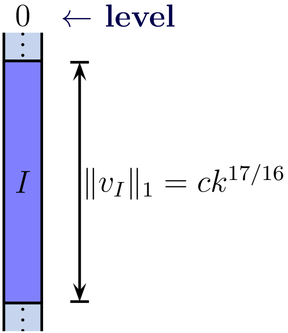
$$v_I = \begin{pmatrix} 1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



- **Given:** x. **Find:** y with $|(\sum_{j \leq i} A_j)(x - y)|$ small
- For interval $I \subseteq [n]$: $v_I := \sum_{i \in I} A_i$

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$



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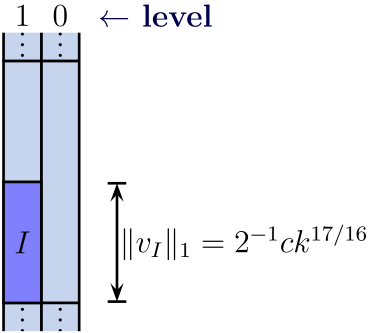
Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$



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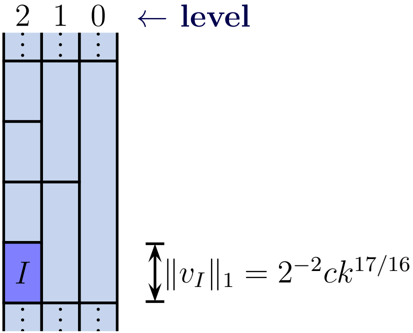
Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$



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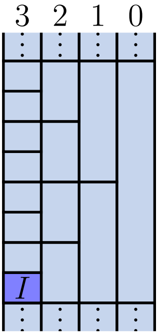
$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$



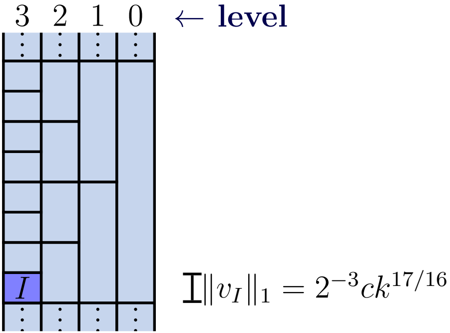
← **level**

$\|v_I\|_1 = 2^{-3}ck^{17/16}$

- **Given:** x. **Find:** y with $|(\sum_{j \le i} A_j)(x - y)|$ small
- For interval $I \subseteq [n]$: $v_I := \sum_{i \in I} A_i$

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

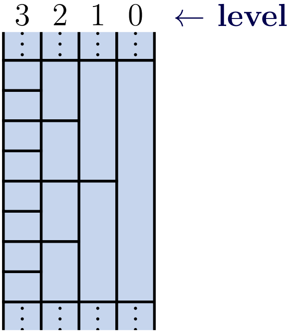


- **Given:** x. **Find:** y with $|(\sum_{j \leq i} A_j)(x - y)|$ small
- For interval $I \subseteq [n]$: $v_I := \sum_{i \in I} A_i$, $\lambda_I :=$ level(I)

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

n columns

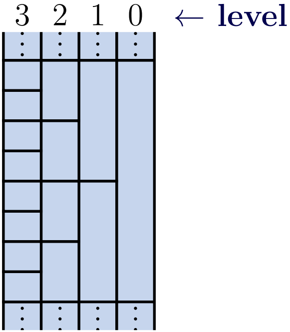


- Run Partial coloring with $v_I := \sum_{i \in I} A_i$ and $\lambda_I := \text{level}(I)$

Applying the Partial Coloring Lemma

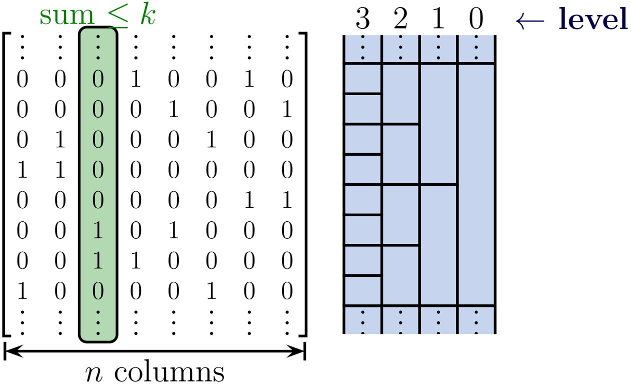
$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

n columns



- Run Partial coloring with $v_I := \sum_{i \in I} A_i$ and $\lambda_I := \text{level}(I)$

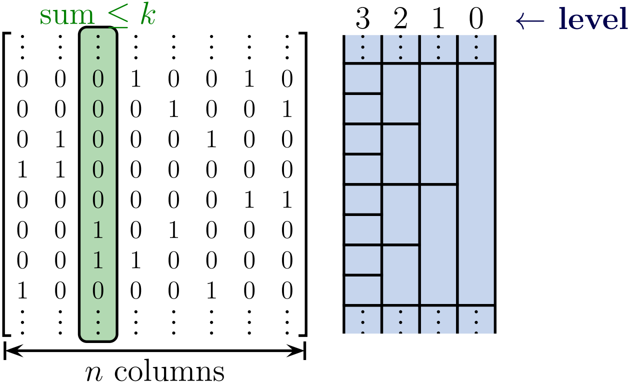
Applying the Partial Coloring Lemma



- Run Partial coloring with $v_I := \sum_{i \in I} A_i$ and $\lambda_I := \text{level}(I)$

$$\sum_I e^{-\lambda_I^2/16}$$

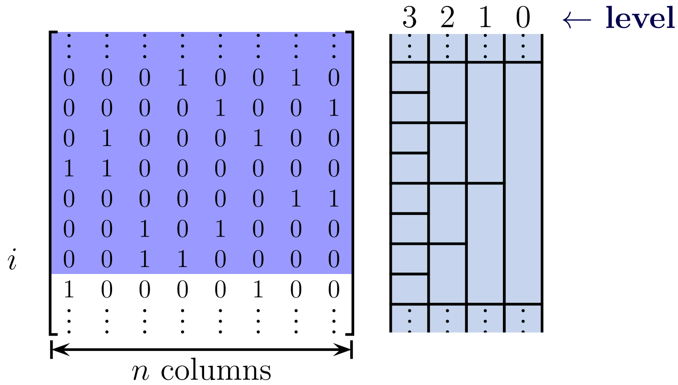
Applying the Partial Coloring Lemma



- Run Partial coloring with $v_I := \sum_{i \in I} A_i$ and $\lambda_I := \text{level}(I)$

$$\sum_I e^{-\lambda_I^2/16} \leq \sum_{\ell \geq 0} \frac{k \cdot n}{ck^{17/16} 2^{-\ell}} \cdot e^{-\ell^2/16}$$

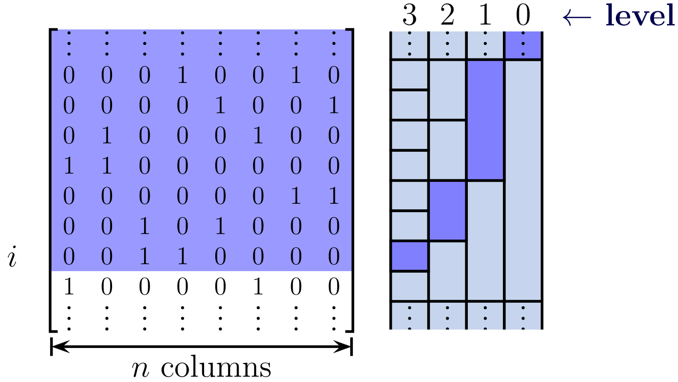
Applying the Partial Coloring Lemma



- Bound error for item i:

$$|(\sum_{j \leq i} A_j)(x - y)| \leq$$

Applying the Partial Coloring Lemma

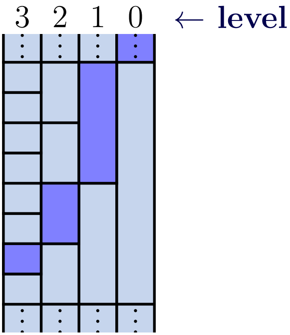


- Bound error for item i:

$$|(\sum_{j \le i} A_j)(x-y)| \le$$

Applying the Partial Coloring Lemma

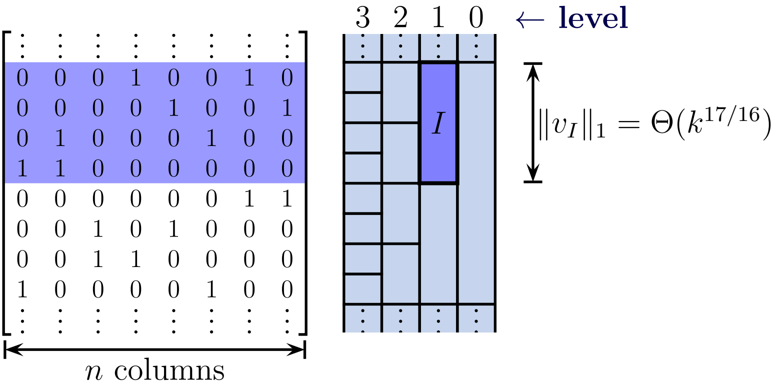
$$\underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{n \text{ columns}}$$



- Bound error for item i:

$$|(\sum_{j \leq i} A_j)(x - y)| \leq \sum_{\ell \geq 0} \ell \cdot \|v_{I \text{ on level } \ell}\|_2$$

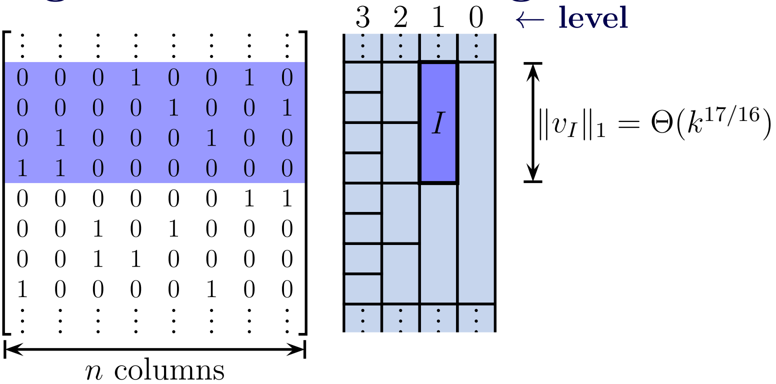
Applying the Partial Coloring Lemma



- Bound error for item i:

$$|(\sum_{j \leq i} A_j)(x - y)| \leq \sum_{\ell \geq 0} \ell \cdot \|v_{I \text{ on level } \ell}\|_2 \leq O(1) \cdot \|v_I\|_2$$

Applying the Partial Coloring Lemma



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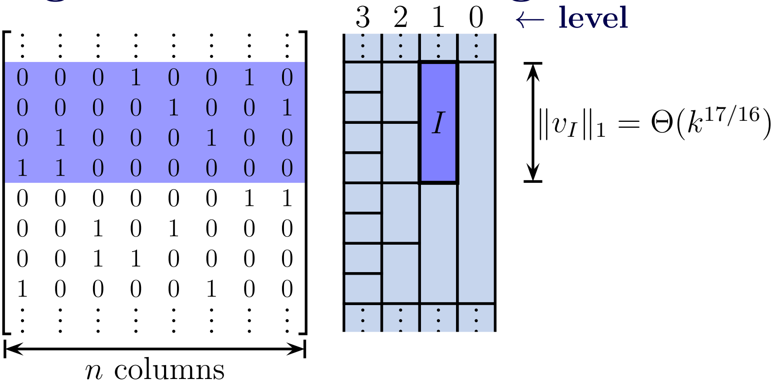
$$\overset{\text{Hölder}}{\le} \quad \|v_I\|_1 \cdot \sqrt{\frac{k^{1/4}}{k^{1/2}}}$$

Applying the Partial Coloring Lemma



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$$\overset{\text{Hölder}}{\le} \quad \|v_I\|_1 \cdot \sqrt{\frac{k^{1/4}}{k^{1/2}}} \le O(k^{15/16})$$

The end

Open problems:

The end

Open problems:

1. Close the gap

$$1 \leq \text{additive integrality gap} \leq O(\log OPT)$$

The end

Open problems:

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$$1 \leq \text{additive integrality gap} \leq O(\log OPT)$$

2. Find more applications in LP rounding.

The end

Open problems:

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Thanks for your attention