An Algorithmic Reduction Theory for Binary Codes

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Joint work with

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Overview

This work

- Propose analogues from lattices to binary codes (Defs, Algs, Bounds)

- ? Use it to speed-up Cryptanalytic algorithms (code-based cryptography)

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This talk

- Recall the LLL alg for Lattices
- Adapt it to Codes

What notion of Orthogonality for binary codes?

Reduction

Find a $\left\{\begin{array}{l}\text{unique}\\ \text{canonical}\\ \text{good}\end{array}\right\}$ representative $x \in X$

of a given class $c \in X/\sim$.

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Lattice Reduction

Find a good basis $B \in GL_n(\mathbb{R})$

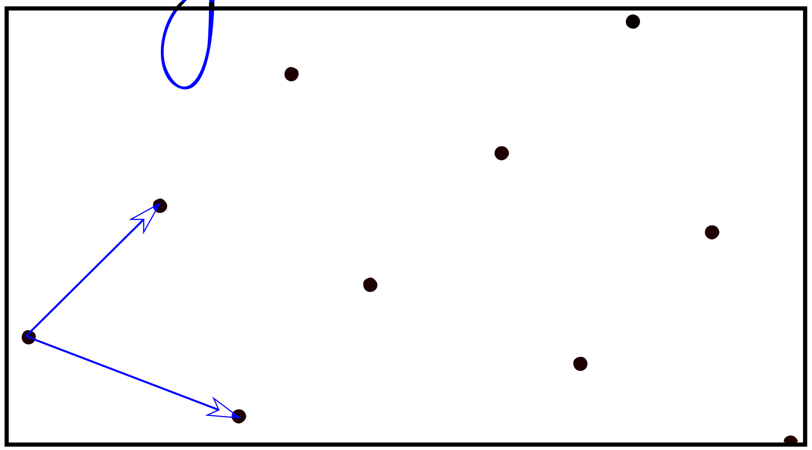
of a lattice $\mathcal{L} \in GL_n(\mathbb{R}) / GL_n(\mathbb{Z})$.

Lattices

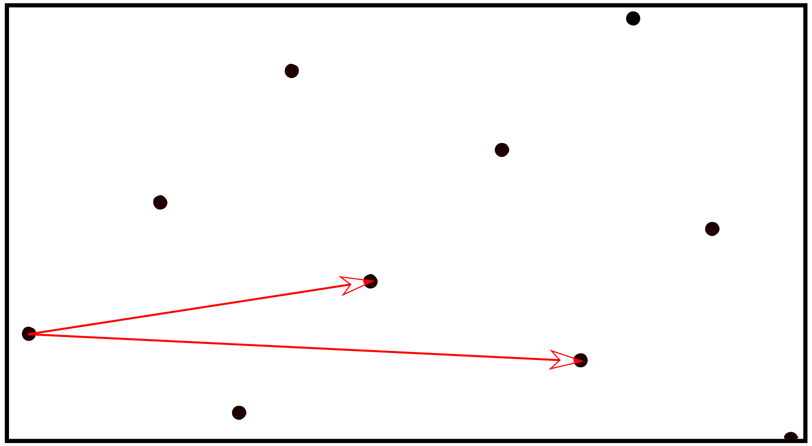
$\mathcal{L} \subseteq \mathbb{R}^n$, a discrete subgroup of a Euclidean Vector Space

Lattices

Good basis



Bad basis



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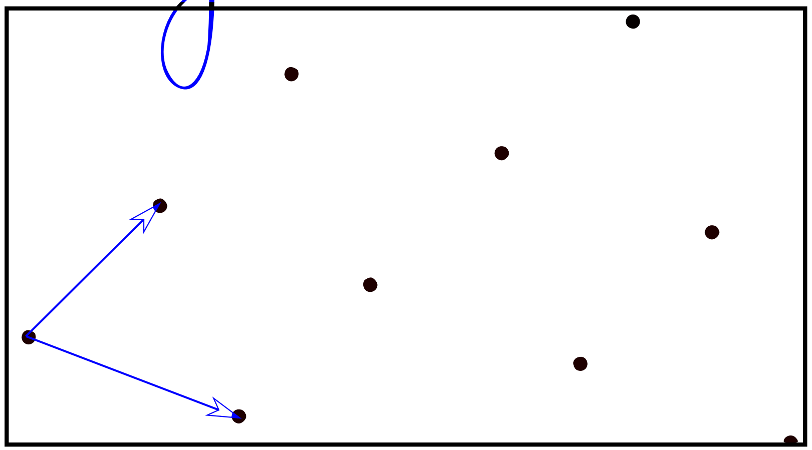

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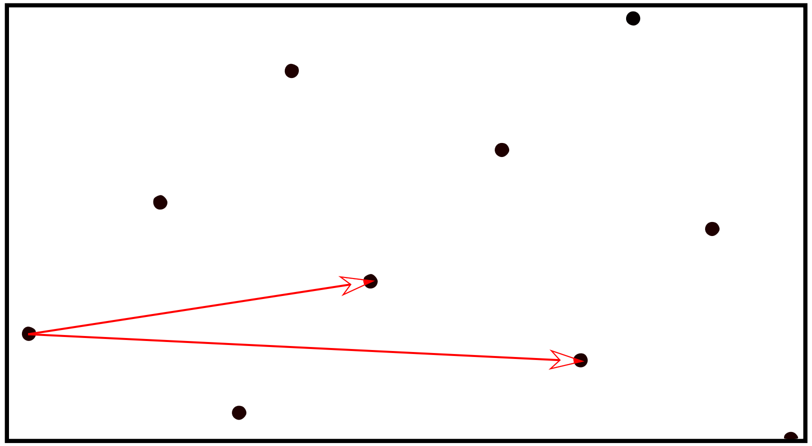
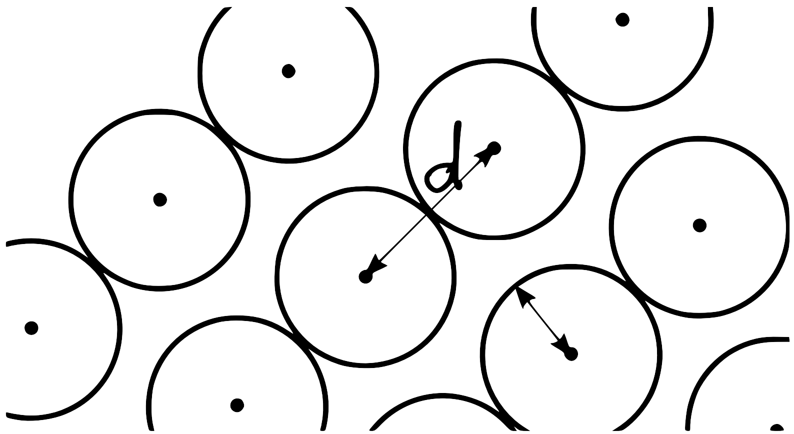


Minimal distance

$$d = \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|$$

$\Longrightarrow$ Sphere packing



Invariants

B and B' generate the same lattice iff :

$$\exists\, U \in GL_n(\mathbb{Z}) \quad \text{st} \quad B' = B \cdot U.$$

$\Rightarrow \quad \det(\mathcal{L}) := \det(B)$ is an invariant of $\mathcal{L}$.

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Gram-Schmidt Orthogonalisation

$$b_i^* := \pi^{\perp}_{(b_1, \dots b_{i-1})}(b_i)$$

$$= b_i - \sum_{j<i} \frac{\langle b_i, b_j^* \rangle}{\langle b_i^*, b_i^* \rangle} \cdot b_j^*$$

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$$\det(\mathcal{L}) = \prod_i \|b_i^*\|$$

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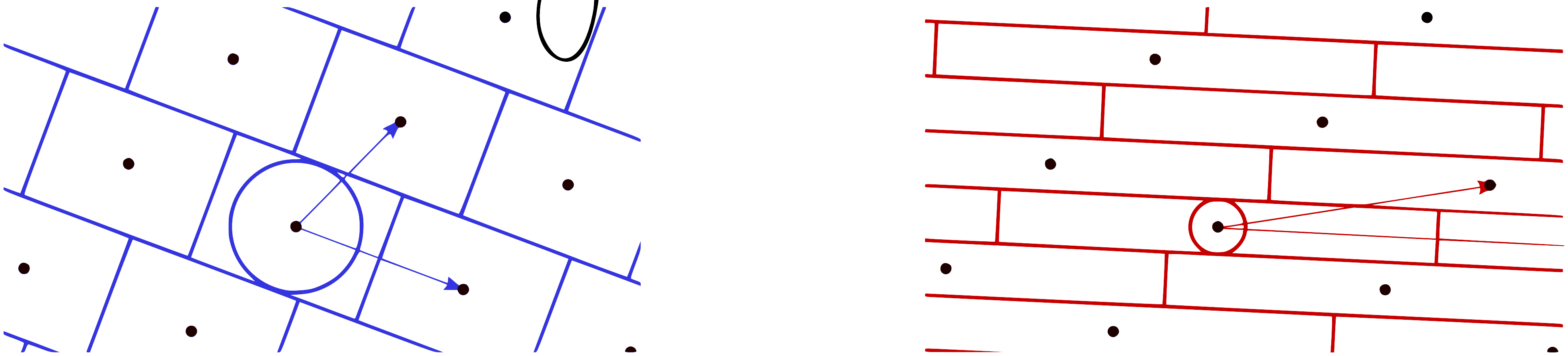
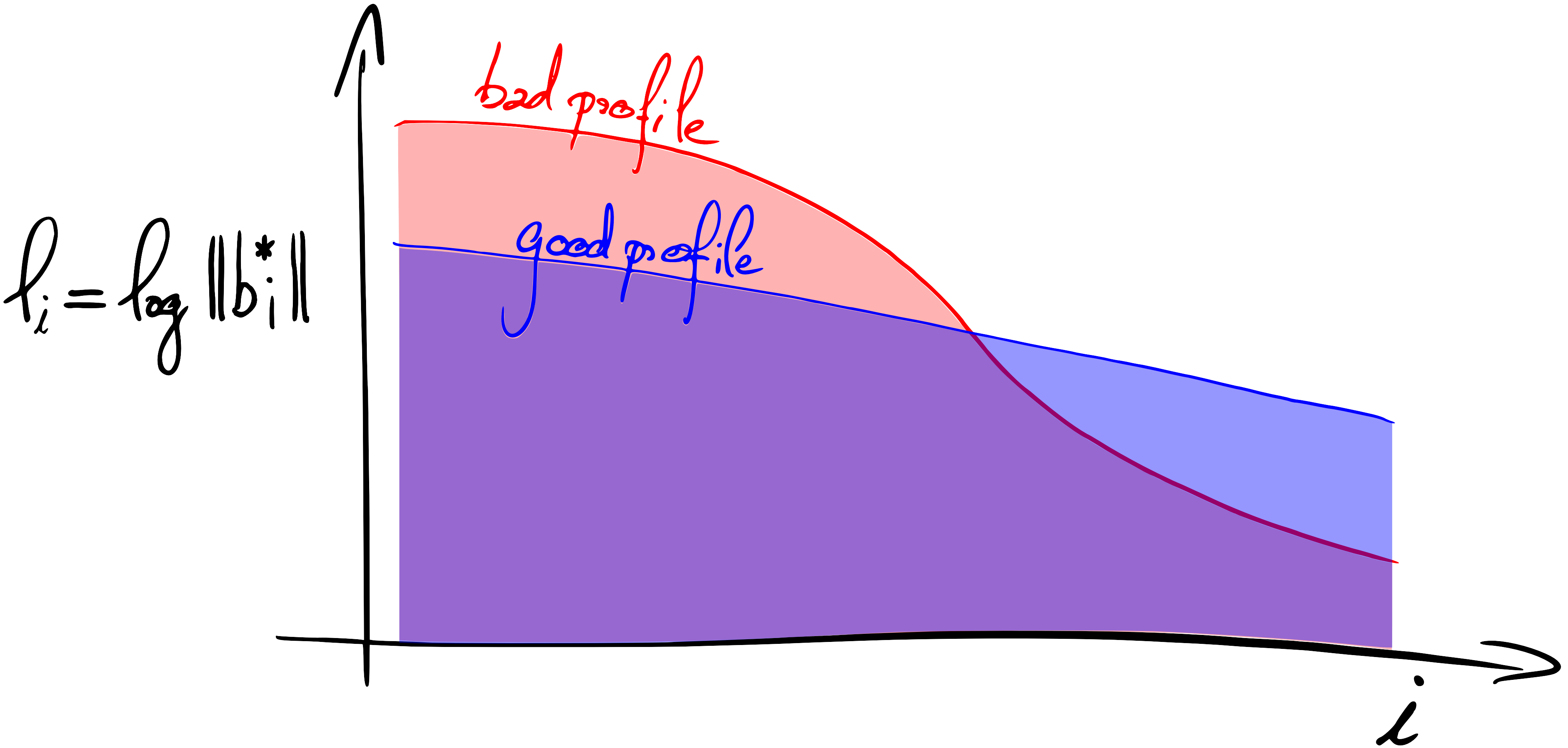


"Good basis" $\iff$ Fundamental Parallelipiped $P(B^*)$ is "close" to a hypercube

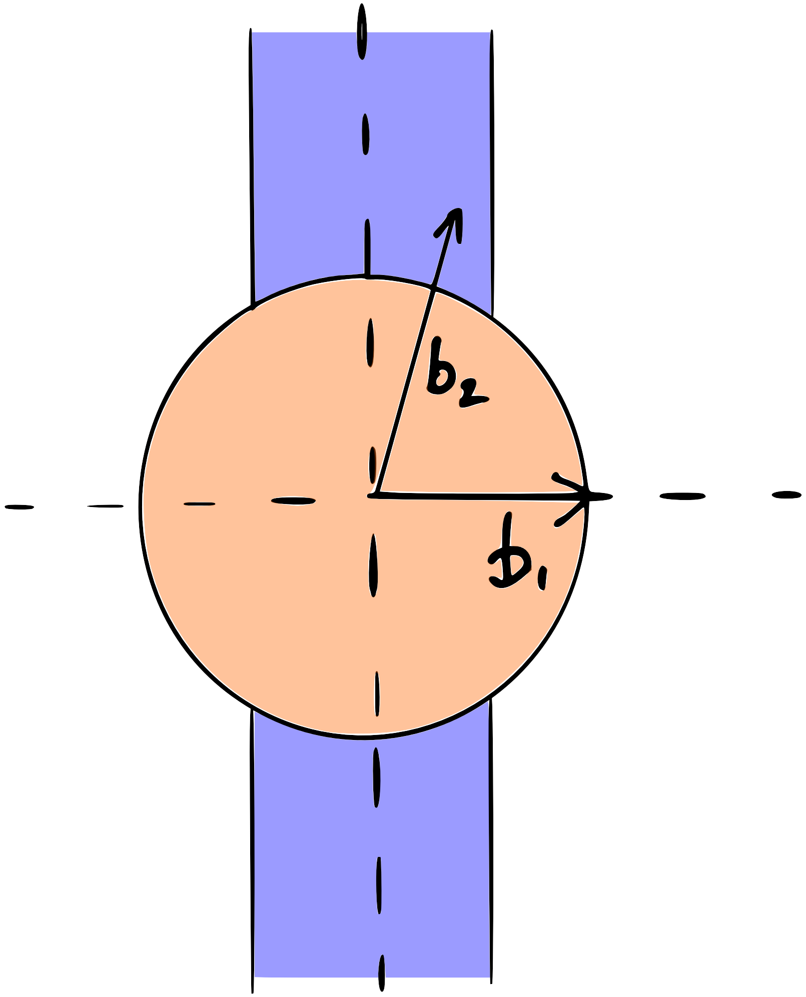
$$\iff \|b_1^*\| \approx \|b_2^*\| \approx \dots \approx \|b_n^*\|.$$

Profile



Area = Area = $\log \det(\mathcal{L})$, invariant.

$n=2$: Lagrange Reduction



Wristwatch Lemma

For any lattice $\mathcal{L}$ of dim 2
$\exists (b_1, b_2)$ a basis s.t.

$$\|b_1\| \leq \|b_2\|$$

$$|\langle b_1, b_2 \rangle| \leq 1/2 \cdot \|b_1\|$$

In particular

$$\|b_1\| \leq \sqrt{4/3} \cdot \|b_2^*\|$$

LLL Reduction

Definition

A basis B of $\mathcal{L}$ is LLL-reduced if $(\pi_i(b_i), \pi_i(b_{i+1}))$ is Lagrange-reduced for all $i < n$.

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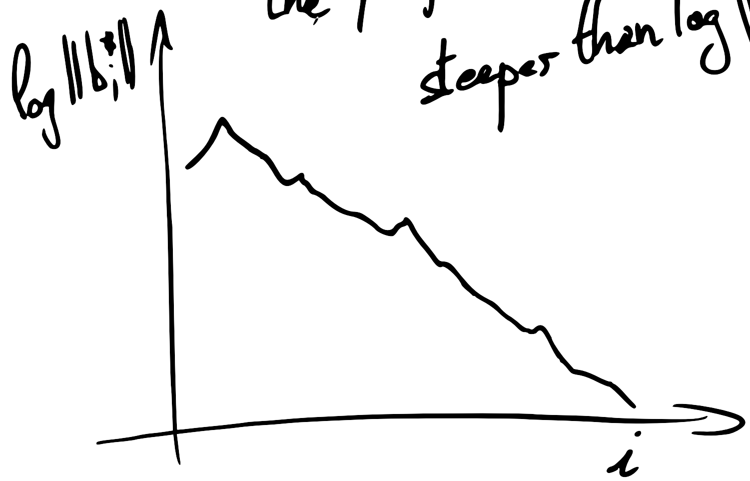
"the profile never falls steeper than $\log\sqrt{4/3}$"



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"the profile never falls steeper than $\log\sqrt{4/3}$"

$$\Longrightarrow \quad \forall i<n,\ \|b_i^*\| \leq \sqrt{4/3}\cdot\|b_{i+1}^*\|$$

Chain & collect $\Longrightarrow$

$$\|b_1\| \leq \sqrt{4/3}^{\frac{n-1}{2}}\cdot\det(\mathcal{L})^{1/n}.$$

LLL Algorithm

While $\exists i$ s.t. $(\pi_i(b_i), \pi_i(b_{i+1}))$ is not *Lagrange-reduced*

Lagrange-reduce it ...

Correctness : Trivial

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Termination in poly-time :

- Requires a slight relaxation (ε-Lagrange-Reduced)
- Proved using a potential argument :

$$P = \sum_{i \leq n} \sum_{j \leq i} \log(\|b_j^*\|)$$

decreases by ε at each step and is lower-bounded.

Binary Codes

$\mathcal{C} \subseteq \mathbb{F}_2^n$ a subspace of a binary vector space, endowed with the Hamming metric.

Binary Codes

Bitstring notation

$x_1 = 0101 \in \mathbb{F}_2^4$ $\quad$ $x_2 = 0011 \in \mathbb{F}_2^4$

XOR (sum): $x_1 \oplus x_2 = 0110$ $\quad$ AND: $x_1 \wedge x_2 = 0001$ $\quad$ OR: $x_1 \vee x_2 = 0111$

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Hamming distance

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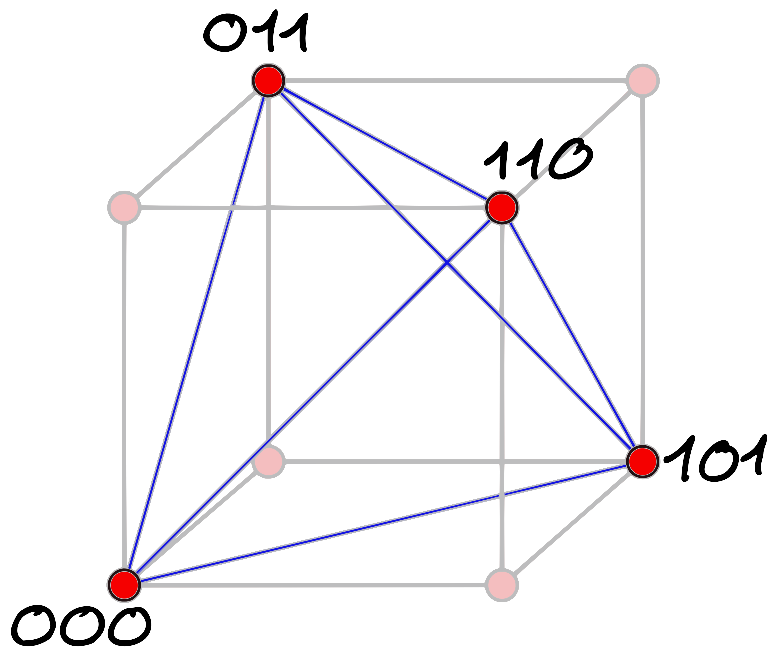

Example

$\mathcal{C} = \{000, 110, 011, 101\}$

$n = 3 \quad k = 2 \quad d = 2$

generated by $b_1 = 110$, $b_2 = 011$

Orthopodality

Inner product

$$\langle x, y\rangle = \sum x_i y_i \mod 2$$

gives no relations on $|x|, |y|$
and $|x \oplus y|$...

Definition $x \perp y$ if

$$Supp(x) \cap Supp(y) = \emptyset$$

$$(eq.\ x \wedge y = 0)$$

$$x \perp y \iff |x \oplus y| = |x| + |y|$$

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Orthopodal projection

$$\pi_x : y \longmapsto y \wedge x$$

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$$\Pi_x(y) \oplus \Pi_x^\perp(y) = y$$

$$\Pi_x(y) \perp \Pi_x^\perp(y)$$

$$|\Pi_x(y)| \leqslant |y|$$

$$\Pi_x^\perp \circ \Pi_y^\perp = \Pi_y^\perp \circ \Pi_x^\perp = \Pi_{x \vee y}^\perp$$

Epipodal matrix

Definition

For a basis $B = b_1, \dots, b_k$, the i-th epipodal vector is defined by

$$b_i^* := \pi^{\perp}_{b_1 \vee b_2 \dots \vee b_{i-1}}(b_i)$$

$$= b_i \wedge \overline{b_1 \vee b_2 \dots \vee b_{i-1}}$$

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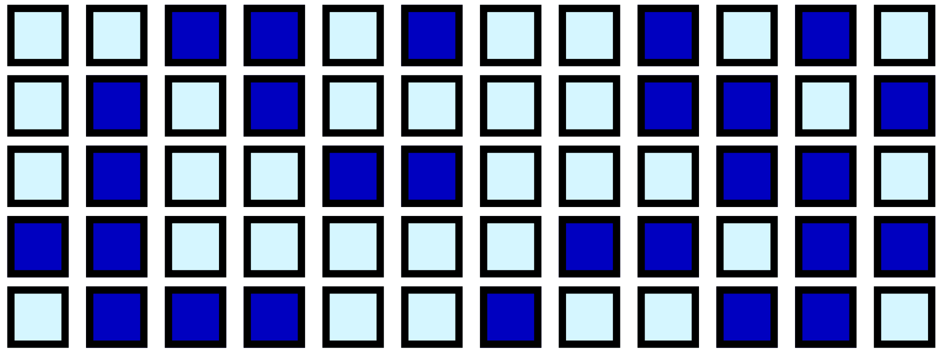

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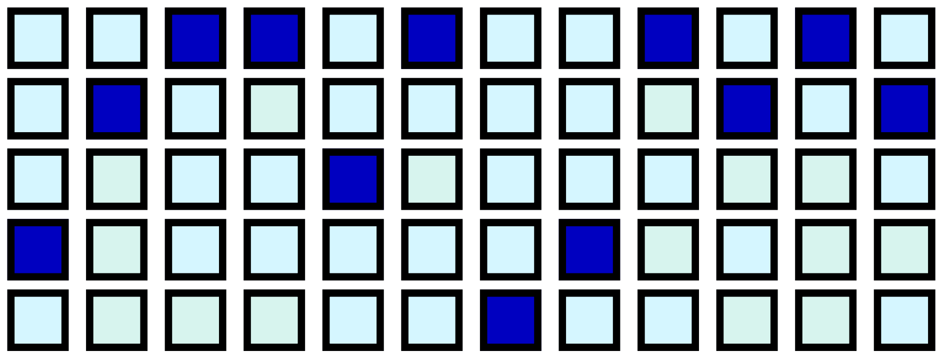

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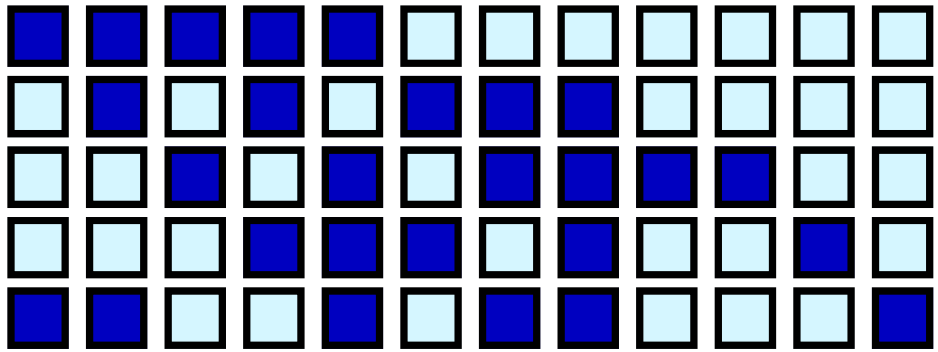

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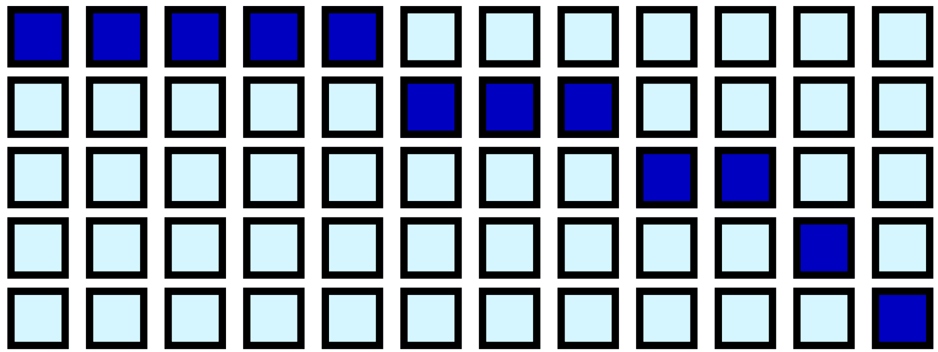

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Gram-Schmidt-like properties

$$\forall i \neq j, \quad b_i^* \perp b_j^*$$

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Invariant

$$\#\mathrm{Supp}(\mathcal{C}) = \sum_i |b_i^*|$$

Analogue to $\det(\mathcal{L}) = \prod \|b_i^*\|$

$k=2$: Mimicking Lagrange

Lemma For any code $\mathcal{C}$ of support size $n = \#\mathrm{Supp}(\mathcal{C})$ and dimension $k=2$, there exist a basis b_1, b_2 s.t.

$$|b_1| \leqslant 2/3 \cdot n \;, \quad |b_2 \wedge b_1| \leqslant 1/2 \cdot |b_1|$$

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Proof $\mathcal{C}$ has 3 distincts non-zero codewords:

Up to isometry (permutation)

$$\begin{array}{rl}
 & \overbrace{\qquad\qquad\quad}^{a}\ \overbrace{\qquad\qquad\quad}^{b}\ \overbrace{\qquad\quad}^{c} \\
c_1 = & \boxed{11 \cdots \quad \cdots \quad \cdots 11}\ \boxed{0 \cdots 00} \\
c_2 = & \boxed{00 \cdots 00}\ \boxed{11 \cdots \quad \cdots \quad \cdots 11} \\
\hline
c_3 = c_1 \oplus c_2 = & \boxed{11 \cdots 11}\ \boxed{00 \cdots 00}\ \boxed{11 \cdots 11}
\end{array}$$

$$\begin{aligned}
|c_1| &= a+b \\
|c_2| &= b+c \\
|c_3| &= a+c \\
n &= a+b+c
\end{aligned}$$

LLL for Codes

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Lagrange-reduce it ...

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Guarantees

$$|b_i^*| \leq 2 \cdot |b_{i+1}^*|, \qquad |b_i^*| \geq 1$$

LLL Bound

Guarentees $\quad \ell_i \leq 2 \cdot \ell_{i+1}, \quad \ell_i \geq 1 \qquad \ell_i := |b_i^*|$

Invariant $\quad \sum_i \ell_i = \# Supp(\mathcal{C}) \leq n$

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$p := \lfloor \log_2 \ell_1 \rfloor \qquad \ell_1 \cdot \sum_{i=0}^{p} 2^{-i} + \sum_{i=p+1}^{k-1} 1 \leq n$

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$$\ell_1 - \left\lfloor \frac{\log_2 \ell_1}{2} \right\rfloor \leq \frac{n-k+1}{2}$$

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griesmer bound

Bounds

		Algorithmic?
Singleton's	$d \leq n-k+1$	Yes
Hamming's	$2^k \cdot \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i} \leq 2^n$	No
Griesmer's	$d - \left\lfloor \frac{\log_2 d}{2} \right\rfloor \leq \frac{n-k+1}{2}$	Yes

What's more

- An analogue to Babai's nearest plane algorithm
- A study of the associated fundamental domains
- A hybrid Lee-Brickell + Babai Algorithm
- Open-source implementation & experiments
- Many open questions, e.g.

What about Duality ?¿

The End

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Pre-print https://eprint.iacr.org/2020/869
Code https://github.com/lducas/CodeRed