Diffusion coefficient of propagating fronts with multiplicative noise

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Recent studies have shown that in the presence of noise, both fronts propagating into a metastable state and so-called pushed fronts propagating into an unstable state, exhibit diffusive wandering about the average position. In this paper, we derive an expression for the effective diffusion coefficient of such fronts, which was motivated before on the basis of a multiple scale ansatz. Our systematic derivation is based on the decomposition of the fluctuating front into a suitably positioned average profile plus fluctuating eigenmodes of the stability operator. While the fluctuations of the front position in this particular decomposition are a Wiener process on all time scales, the fluctuations about the time-averaged front profile relax exponentially.

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I. INTRODUCTION

One of the aspects of front propagation that have been studied in the literature in recent years is the effect of fluctuations on propagating fronts [1–4]. In particular, it has been found that in the presence of noise, both one-dimensional fronts between a stable and a metastable state ("bistable fronts") and so-called *pushed* fronts, which propagate into an unstable state, [5], exhibit a diffusive wandering about their average position [4]. This contrasts with the fluctuation behavior of so-called *pulled* fronts propagating into an unstable state which is subdiffusive [6]. In this paper, we shall consider only the case of pushed and bistable fronts, however.

Recently, Armero *et al.* [4] derived an expression for the effective diffusion coefficient of a pushed front in the stochastic field equation

$$\frac{\partial \phi}{\partial t}=\frac{\partial^2 \phi}{\partial x^2}+f(\phi)+\varepsilon^{1/2}g(\phi)\,\eta(x,t) \tag{1}$$

with a noise term whose correlations are

$$\overline{\eta(x,t)}=0, \tag{2}$$

$$\overline{\eta(x,t)\,\eta(x',t')}=2C(|x-x'|/\Lambda)\,\delta(t-t'). \tag{3}$$

In Eq. (1), f is a nonlinear function of the field ϕ with a stable state at $\phi=1$ and either a (meta)stable or unstable state at $\phi=0$ and $g(\phi)$ is some other general nonlinear function. In Eqs. (2) and (3), the overbar denotes an average over the realizations of the noise. In order that our noise of Stratonovich type is well defined, we have introduced a spatial cutoff in the noise correlation function (3) (see [4] for further details).

The derivation in [4] of the effective front diffusion coefficient D_f relied on a small-noise stochastic multiple-scale analysis that was based on the idea that the mean-square displacement of the front about its average position was slow relative to the deterministic relaxation of the front. The basic idea was that only the low-frequency components of the noise are responsible for the front wandering, so that the high-frequency components, which renormalize the front shape and its velocity, could be implicitly integrated out. This led to an ansatz for the relative scaling of fast and slow time variables where the small parameter governing the separation of time scales was the diffusion coefficient D_f of the front itself. The method then self consistently provided an explicit prediction for D_f, which was in good agreement with their numerical results. The main weakness of the approach was that the above coarse-graining procedure could not be carried out explicitly, since while there is a separation of time scales for the *average* quantities, a scale separation scheme is not natural for the *fluctuating* quantities. Hence, the derivation had to rely on an uncontrolled ansatz. In this brief report, we therefore reconsider this problem. We justify the previously derived result for D_f with a systematic small-noise expansion based on decomposing the motion of the front into a diffusive motion of the properly defined front position, plus fluctuations about the average front profile. Technically, the fluctuating front position is defined by requiring that the fluctuations about the mean front profile are orthogonal to the (left) translation mode. This derivation shows that the previous multiple-scale ansatz is not quite adequate, and it will clarify the connection between the separation of time scales invoked in Ref. [4], the small-noise expansion, and the existence of a finite gap in the linearized evolution operator. The key point of our fully systematic derivation is the fact that there is a unique choice for the collective coordinate $X(t)$ of the front profile to be a memory-less Markovian process, and that the fluctuations about the average profile then relax exponentially. This relaxation may be deduced from the spectrum of the linearization operator about the average front profile. In addition, our method provides a general strategy to address the problem of fluctuations of fronts and other coherent structures, and may be extended to higher-order perturbation theory.

II. DERIVATION OF THE EFFECTIVE DIFFUSION COEFFICIENT

We may rewrite Eq. (1) in terms of a noise term R whose average $\overline{R}$ is zero and a deterministic renormalized part,

$$\frac{\partial \phi}{\partial t}=\frac{\partial^2 \phi}{\partial x^2}+h(\phi)+\varepsilon^{1/2}R(\phi,x,t), \tag{4}$$

using Novikov's Theorem, as discussed in [4]. In Eq. (4),

$$h(\phi)=f(\phi)+\varepsilon C(0)g'(\phi)g(\phi), \tag{5}$$

$$R(\phi,x,t)=g(\phi)\eta(x,t)-\varepsilon^{1/2}C(0)g'(\phi)g(\phi), \tag{6}$$

where $C(0)$ is of order Λ^{-1}, so that Eq. (3) yields a delta correlation in space in the limit $\Lambda\to 0$ [7]. The main idea of the derivation is to introduce a collective coordinate $X(t)$ for the position of the front. Of course, there are various choices for the position $X(t)$, but as we shall show a particular choice makes the equations quite transparent. We decompose the fluctuating field ϕ as

$$\phi=\phi_0[\xi-X(t)]+\phi_1[\xi-X(t),t]. \tag{7}$$

Here, ϕ_0 is the solution of the ordinary differential equation for the shape of a deterministic front with velocity v_R, the velocity of the deterministic front associated with Eq. (4) with $R=0$ (the subscript R on v_R reminds us that the front speed is determined by $h(\phi)$ rather than $f(\phi)$, and hence, is renormalized due to the presence of the noise). In other words, ϕ_0 satisfies

$$0=\frac{d^2\phi_0(\xi)}{d\xi^2}+v_R\frac{\partial \phi_0(\xi)}{\partial \xi}+h(\phi_0). \tag{8}$$

While ϕ_0 is a nonfluctuating quantity, ϕ_1 is a stochastic field that contains the fluctuations about ϕ_0. In the above, $\xi=x-v_Rt$ is the proper variable for a deterministic front moving with the asymptotic velocity v_R, but note that in Eq. (7), the fields are written in terms of the shifted variable

$$\xi_X=\xi-X(t)=x-v_Rt-X(t), \tag{9}$$

where $X(t)$ is the rapidly fluctuating front position whose explicit definition in terms of a spatially averaged front profile is given below.

As is well known, the derivation of a moving boundary approximation for deterministic equations (see, e.g., [8,9] and references therein) normally proceeds by projecting onto the zero mode. Indeed, associated with the front solution ϕ_0 of Eq. (8) is a zero mode of the stability operator

$$\mathcal{L}=\frac{\partial^2}{\partial \xi^2}+v_R\frac{\partial}{\partial \xi}+h'(\phi_0), \tag{10}$$

which is obtained by linearizing about ϕ_0. This zero mode expresses translational invariance, and indeed implies that

$$\mathcal{L}\Phi_R^{(0)}=0\Leftrightarrow\Phi_R^{(0)}=\frac{d\phi_0}{d\xi}. \tag{11}$$

In our case the operator $\mathcal{L}$ is not self adjoint, since $v_R\neq 0$; as a result, the left eigenmode $\Phi_L^{(0)}$ is different from $\Phi_R^{(0)}$, but it is known to be (see, e.g. [4,9])

$$\mathcal{L}^+\Phi_L^{(0)}=0\Leftrightarrow\Phi_L^{(0)}=e^{v_R\xi}\frac{d\phi_0}{d\xi}. \tag{12}$$

As we mentioned above, a particular definition of the position $X(t)$ is especially convenient: we take $X(t)$ defined implicitly by the requirement that the fluctuating field ϕ_1 is orthogonal to the left zero mode. Indeed, defining

$$\langle A(\xi)B(\xi)\rangle=\int_{-\infty}^{\infty}d\xi A(\xi)B(\xi), \tag{13}$$

we require

$$\langle\Phi_L^{(0)}\phi_1(\xi,t)\rangle=\int d\xi e^{v_R\xi}\frac{d\phi_0}{d\xi}[\phi-\phi_0(\xi-X(t))]=0. \tag{14}$$

Note that at any moment, the *fluctuating* front position $X(t)$ is defined in terms of *weighted spatial average* of the fluctuating field ϕ.

Upon substitution of Eq. (7) into Eq. (4) and linearization in ϕ_1 (which is justified for small noise), we obtain

$$\frac{\partial \phi_1}{\partial t}=\mathcal{L}\phi_1-\dot{X}(t)\frac{\partial \phi_0}{\partial \xi_X}+R(\phi_0,\xi,t). \tag{15}$$

Note that we have also approximated $R(\phi,\xi_X,t)$ by $R(\phi_0,\xi,t)$, which again is correct to lowest order in the noise.

In addition to the zero mode, the operator $\mathcal{L}$ will in general have right eigenmodes $\Phi_R^{(l)}$ with eigenvalues $-\sigma_l$:

$$\mathcal{L}\Phi_R^{(l)}=-\sigma_l\Phi_R^{(l)},\quad l\neq 0, \tag{16}$$

and with associated left eigenfunctions $\Phi_L^{(l)}=e^{v_R\xi}\Phi_R^{(l)}$. Our convention to have the eigenvalues $-\sigma_l$ anticipates that the dynamically relevant front solution is stable, so that all eigenvalues σ_l are positive. Moreover, both for fronts between a stable and a metastable state and for pushed fronts propagating into an unstable state, the spectrum is known to be gapped [10,11], i.e., the smallest eigenvalue is strictly greater than zero [10,11].

Since ϕ_1 is orthogonal to $\Phi_L^{(0)}$, we can expand ϕ_1 in terms of the eigenmodes $\Phi_R^{(l)}$ $(l\geqslant 1)$ of $\mathcal{L}$ as

$$\phi_1(\xi_X,t)=\sum_{l\neq 0}a_l(t)\Phi_R^{(l)}(\xi_X). \tag{17}$$

Substitution of this expansion into Eq. (15) then yields upon projection onto the zero mode $\Phi_L^{(0)}$:

$$\dot{X}(t)=\varepsilon^{1/2}\frac{\langle\Phi_L^{(0)}R(\phi_0,\xi,t)\rangle}{\langle\Phi_L^{(0)}\Phi_R^{(0)}\rangle}. \tag{18}$$

Taking the square of this result, integrating and averaging over the noise,

$$\overline{X^2(t)}=2D_f t=\int_0^t dt'\int_0^t dt''\overline{\dot{X}(t')\dot{X}(t'')},\tag{19}$$

then yields with Eqs. (3), (11), and (12)

$$D_f=\varepsilon\frac{\int d\xi\, e^{2v_R\xi}(d\phi_0/d\xi)^2 g^2(\phi_0)}{\left[\int d\xi\, e^{v_R\xi}(d\phi_0/d\xi)^2\right]^2}.\tag{20}$$

This is precisely the result derived earlier in [4], but now in a fully systematic way. To lowest order in the present small-noise expansion, the average front profile is simply ϕ_0. However, notice that ϕ_0 contains a dependence on Λ through $C(0)$ in $h(\phi)$. The parameter $C(0)$ must be considered as an independent one, so that the result (20) has to be interpreted as to first order in ε but to all orders in ε/Λ.

The above derivation allows us to also obtain the relaxation of a fluctuation about the average. Indeed, upon substituting Eq. (17) into Eq. (15) and projecting onto the left zero modes, using $\langle\Phi_L^{(n)}\Phi_R^{(m)}\rangle=\delta_{nm}$ for normalized eigenmodes, we obtain to lowest order

$$\frac{da_l}{dt}=-\sigma_l a_l+\varepsilon^{1/2}\langle\Phi_L^{(l)}R\rangle,\tag{21}$$

as terms $\dot{X}(t)d\phi_1/d\xi$ are of higher order in ε. Note that each mode is damped and has its noise strength weighted by $\Phi_L^{(l)}$. One may derive from here in a straightforward way the mean square fluctuations about the average profile.

We finally note that our discussion clarifies the difficulty of using a separation of time scales argument for the derivation of the effective diffusion coefficient: the collective coordinate $X(t)$ is a memory-less Markov process, and hence, the changes in the position have zero correlation time while the average of $X^2(t)$ changes slowly. The coefficients $a_l(t)$, on the other hand, have a finite correlation time, and hence, are correlated on timescales in between the one of instantaneous position $X(t)$ and the mean-square wandering $\overline{X^2(t)}$.

III. CONCLUDING REMARKS

We have reported an improved derivation of the diffusion coefficient of propagating pushed fronts with multiplicative noise, previously found in Ref. [4]. The present derivation is fully explicit and based on standard projection techniques. The key point is the identification of a definition of the front position, which naturally implies the diffusive wandering of the front, and avoids invoking an uncontrolled hypothesis in addition to the basic assumption of small-noise strength. This has also clarified that the time scale separation used in Ref. [4] may be traced back to the small-noise approximation together with the existence of a finite gap in the spectrum of the linearized evolution operator. All these considerations may be generalized to the effect of fluctuations on other types of coherent structures.

Our derivation of the solvability expression (20) for D_f of a propagating front shows that the collective coordinate $X(t)$ responds instantaneously to the noise R: There is no memory term in (18), so that $X(t)$ is Markovian and, more precisely, it coincides with the Wiener process (to lowest order in the noise strength). We stress that this is only true for our particular definition of $X(t)$ in terms of the orthogonality of ϕ_1 to the left zero mode. For any other definition, such as the usual one to define the front position as $X(t)=\int d\xi\,\phi(\xi)$, $X(t)$ will not be a Markov process, and would show only diffusive behavior at sufficiently long time scales.

As a byproduct of our derivation, we have also obtained an explicit expression for the relaxation behavior of the fluctuations about the mean front profile. Not surprisingly, the larger the gap in the spectrum, the faster the relaxation. As is well known, in models in which there is a transition from the pushed regime to the pulled regime, the gap closes upon approaching the transition from the pushed side [10]. Hence, the relaxation becomes slower and slower. As is discussed in [10], in the pulled regime, the spectrum is gapless and this leads to anomalous power-law relaxation of deterministic fronts towards their asymptotic speed and shape. As a result, pulled fronts cannot be described by a moving boundary approximation [9] and in the presence of fluctuations, they exhibit subdiffusive wandering [6] in one dimension and anomalous scaling in higher dimensions [12,13].

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