Universal Algebraic Relaxation of Fronts Propagating into an Unstable State and Implications for Moving Boundary Approximations

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We analyze the relaxation of fronts propagating into unstable states. While "pushed" fronts relax exponentially like fronts propagating into a metastable state, "pulled" or "linear marginal stability" fronts relax algebraically. As a result, for thin fronts of this type, the standard moving boundary approximation fails. The leading relaxation terms for velocity and shape are of order $1/t$ and $1/t^{3/2}$. These universal terms are calculated exactly with a new systematic analysis that unifies various heuristic approaches to front propagation. [S0031-9007(98)05413-1]

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Consider propagating fronts in systems with a continuous order parameter, where a stable state invades an *unstable* state, and assume, that thermal perturbations can be neglected. Such fronts arise in many convective instabilities in fluid dynamics such as in the wake of bluff bodies [1], in Taylor [2] and Rayleigh-Bénard [3] convection, they play a role in spinodal decomposition near a wall [4], the pearling instability of laser-tweezed membranes [5], the formation of kinetic, transient microstructures in structural phase transitions [6], dielectric breakdown fronts [7], the propagation of a superconducting front into a normal metal [8], or in error propagation in extended chaotic systems [9]. For such front propagation problems, it is known [10–14] that if the initial profile is steep enough, arising, e.g., through a local initial perturbation, the propagating front in practice always relaxes to a unique shape and velocity. Depending on the nonlinearities, one can distinguish two regimes: As a rule, fronts whose propagation is driven (pushed) by the nonlinearities very much resemble fronts propagating into metastable states. This regime is often referred to as "pushed" [10,14] or "nonlinear marginal stability" [13]. If, on the other hand, nonlinearities mainly cause saturation, fronts propagate with a velocity determined by linearization about the unstable state: it is as if they are pulled by the linear instability ("pulled" [10,14] or "linear marginal stability" [13] regime). Some heuristic arguments have been put forward [13] that for large times t the velocity and shape of a pulled front generally relax slowly, as $1/t$. The experimental relevance of such a slow relaxation is illustrated by propagating Taylor vortex fronts. Here the measured front velocities [2] were about 40% lower than predicted theoretically; only later it became clear [15] that this was due to slow transients.

In this paper, we identify the general mechanism leading to slow relaxation of uniformly translating fronts, use it to introduce a systematic analysis which allows us to determine *all universal asymptotic terms*, and point out the implication of the relaxation for the existence of a moving boundary approximation.

Our present investigation was, in fact, motivated by an attempt to derive a moving boundary approximation for a thin front propagating into an unstable state [7]. Moving boundary approximations have been applied quite successfully to patterns consisting of domains where the order parameter field varies slowly in space and time due to the coupling to some external field (e.g., temperature in a solidification or combustion front), separated by thin interfacial zones where the order parameter field varies rapidly [16]. Implicit in this method is the assumption that the dynamics on the "inner" interfacial scale and the "outer" pattern scale is adiabatically decoupled in the thin interface limit, so that the boundary conditions for the motion of the interface on the outer scale are *local in space and time*. However, we find that when the moving boundary amounts to a "pulled" front propagating into an unstable state, the standard moving boundary approximation *breaks down*. The physical reason is simply that due to the algebraic relaxation on the inner scale, the time scales for the dynamics on the inner and outer scales are not adiabatically decoupled. As we will discuss in detail elsewhere [17], technically the analysis breaks down due to the nonexistence of solvability integrals associated with the same linear operator $\mathcal{L}^*$ below that plays a role in the relaxation analysis of pulled fronts. "Pushed" fronts, on the other hand, relax exponentially to an asymptotic shape and velocity in much the same way as fronts propagating into a metastable state do. The important distinction for the validity of a moving boundary approximation is thus between pulled fronts on the one hand, and pushed fronts or fronts propagating into a metastable state on the other.

The new approach that we introduce here grew out of studying the above question, and allows us to determine both the velocity and the shape relaxation of pulled fronts systematically. We are able to calculate all universal terms in an asymptotic long time expansion explicitly and exactly, and confirm our predictions numerically. Besides being of interest in their own right, our results identify the general mechanism that leads to the slow relaxation of sufficiently steep initial conditions towards the "pulled" or "marginally stable" front and the concomitant breakdown of a the standard moving boundary approximation; in addition, the analysis welds various seemingly

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unrelated and often heuristic approaches [12–14,18–20] together into a systematic calculational framework with new predictive power.

With "universal" we mean that not only the asymptotic profile is unique, but also the relaxation towards it, provided we start with sufficiently steep initial conditions. This is analogous to the universal corrections to scaling in critical phenomena, if we think of the relaxation as the approach to a unique fixed point in function space along a unique trajectory. The universal velocity and shape relaxation terms which we calculate exactly are of order $1/t$ and $1/t^{3/2}$. The next term in the long time expansion, which is of $O(1/t^2)$, is affected by a time translation $t \to t + t_0$ in the $1/t$ term. The $1/t^2$ terms therefore depend on the initial conditions.

Our analysis can be formulated quite generally for partial differential equations which are of first order in time but of arbitrary order in space, as long as they admit uniformly translating pulled solutions, as defined below. For ease of presentation we guide our discussion along two examples which we have investigated analytically as well as numerically. Our first example is the prototype nonlinear diffusion equation,

$$\partial_t \phi(x,t) = \partial_x^2 \phi + f(\phi), \qquad f(\phi) = \phi - \phi^3, \quad (1)$$

with ϕ, x, t real. This equation is also known as KPP equation (after Kolmogorov *et al.*), Fisher equation, or FK equation. In (1), the state $\phi = 0$ is unstable and the states $\phi = \pm 1$ are stable. We consider a situation where initially $\phi(x,0)$ asymptotically decays quicker than e^{-x} for large x, or, in particular, one with $\phi(x,0) \neq 0$ in a localized region only. The region with $\phi \neq 0$ expands in time, and a propagating front evolves. It has been proven rigorously, that relaxation is always to a unique front profile $\phi^*(x - v^* t)$ with velocity $v^* = 2$ [11], and that the velocity relaxes asymptotically as $v(t) = 2 - 3/(2t)$ [19]. Our second example is the "EFK" (extended FK) equation

$$\begin{aligned} \partial_t \phi(x,t) &= \partial_x^2 \phi - \gamma \partial_x^4 \phi + f(\phi), \\ f(\phi) &= \phi - \phi^3, \end{aligned} \quad (2)$$

which serves as a model for equations with higher spatial derivatives. For $0 \leq \gamma < 1/12$, sufficiently steep initial conditions also evolve into a pulled front translating uniformly with velocity v^* (3) [13,21], but the rigorous methods of [11,19] are not applicable here.

Since the basic state $\phi = 0$ into which the front propagates is linearly unstable, even a small perturbation around $\phi = 0$ grows and spreads by itself. According to the *linearized* equations any localized small perturbation will spread asymptotically for large times with the linear marginal stability speed v^*. This speed is determined explicitly by the linear dispersion relation $\omega(k)$ of a Fourier mode $e^{-i\omega t + ikx}$ [12,13,18] through

$$\left.\frac{\partial \,\mathrm{Im}\,\omega}{\partial \,\mathrm{Im}\,k}\right|_{k^*} - v^* = 0, \qquad \left.\frac{\partial \,\mathrm{Im}\,\omega}{\partial \,\mathrm{Re}\,k}\right|_{k^*} = 0, \qquad \frac{\mathrm{Im}\,\omega(k^*)}{\mathrm{Im}\,k^*} = v^*. \quad (3)$$

The first two equations in (3) are saddle point equations in the complex k plane that govern the long time asymptotics of the Green's function in a frame moving with the leading edge of the front. The third equation expresses that for self-consistency, the linear part of the front should neither grow nor decay in the co-moving frame. If in the full nonlinear equation a front with velocity v^* is unstable or nonexistent, the marginally stable front with velocity $v^\dagger > v^*$ is called "nonlinearly marginally stable" or *pushed*. If a front propagating with velocity v^* is stable, it is called "linearly marginally stable" or *pulled* [10].

Our relaxation analysis applies in general to equations, in which a front solution ϕ^* propagating with velocity v^* (3) is uniformly translating $[\mathrm{Re}\,k^* = 0 = \mathrm{Re}\,\omega(k^*)]$ and dynamically stable, and to all initial conditions that are sufficiently steep in the sense that $\lim_{x\to\infty} \phi(x,0)e^{\Lambda x} = 0$, where $\Lambda = \mathrm{Im}\,k^* > 0$.

We now first summarize our predictions: If we trace the velocity $v_h(t) = \dot{x}_h(t)$ of a fixed amplitude h, where $\phi(x_h(t),t) = h$, we find $v_h(t) = v^* + \dot{X}(t) + g(h)/t^2$ with

$$\dot{X}(t) = \frac{-3}{2\Lambda t}\left(1 - \frac{\sqrt{\pi}}{\Lambda\sqrt{Dt}}\right), \quad (4)$$

in fact independent of h and of the precise initial conditions. Here $D = \frac{1}{2}\partial^2\,\mathrm{Im}\,\omega/(\partial\,\mathrm{Im}\,k)^2|_{k^*}$ plays the role of a diffusion coefficient. The leading $1/t$ term reproduces Bramson's exact result [19] for Eq. (1). Note that all terms in (4) depend on the linear dispersion relation only.

The velocity of the relaxing front is smaller than that of the asymptotic uniformly translating front. The correction is $\dot{X}(t) \approx -3/(2\Lambda t)$ to dominant order. This means that the distance between the asymptotic and the relaxing front grows logarithmically in time as $X(t) \approx -3/(2\Lambda)\ln t$. Since the front width is *finite* in equations like (1) and (2), while $X(t)$ diverges, this immediately explains why the leading velocity correction has to be the same for *all* values of the amplitude h.

If we want to write the shape of the transient front as a small perturbation η about the asymptotic shape ϕ^* at all times, we *have to* linearize about the asymptotic profile $\phi^*(x - v^* t - X(t))$ translated with the nonasymptotic speed $v(t) = v^* + \dot{X}$. This is a crucial ingredient of our analysis. Indeed, when written in the frame $\xi = x - v^* t - X(t)$, we find through an expansion in the "interior region" of the front, where $|\eta| \ll \phi^*$:

$$\begin{aligned} \phi(\xi,t) &= \phi^*(\xi) + \eta(\xi,t), \quad \text{with } \xi = x - v^* t - X(t) \quad (5) \\ &= \phi^*(\xi) + \dot{X}(t)\eta_{\mathrm{sh}}(\xi) + O(t^{-2}), \quad \text{where } \eta_{\mathrm{sh}} = (\delta\phi_v/\delta v)|_{v^*} \\ &= \phi_{v(t)}(\xi) + O(t^{-2}), \quad \text{for } \xi \ll 2\sqrt{Dt}\,. \quad (6) \end{aligned}$$

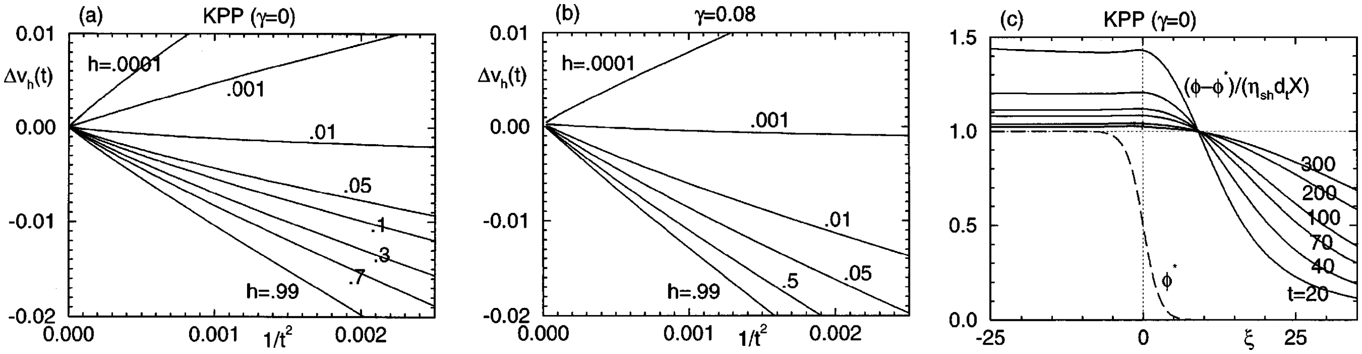

FIG. 1. (a) and (b): Velocity correction $\Delta v_h(t) = v_h(t) - v^* - \dot{X}$ as a function of $1/t^2$ for various amplitudes $\phi(x_h, t) = h$, $v_h = \dot{x}_h$ and for $t \geq 20$. (a): Equation (1), thus $\gamma = 0$, $\Lambda = 1 = D$, and $v^* = 2$. (b): Equation (2) with $\gamma = 0.08$, thus $D = 0.2$, $\Lambda = 1.29$, and $v^* = 1.89$. (c): Data from (a) plotted as $[\phi(\xi, t) - \phi^*(\xi)]/[\dot{X}\eta_{\rm sh}(\xi)]$ over ξ for various t. $\phi^*(\xi)$ (dashed curve) for comparison.

Here ϕ_v is the shape of a front propagating uniformly with velocity v, so Eq. (6) expresses that for large times, the shape of the profile is to a good approximation given by the uniformly translating solution $\phi_v(\xi)$ with the *instantaneous* value of the velocity $v(t) < v^*$. Based on numerical observations, Powell *et al.* [20] have conjectured such a form for the transient profile for equations of type (1). Here it comes out naturally from our general analysis, together with an explicit expression for $X(t)$. Moreover, we find nonvanishing corrections in order $1/t^2$.

In the far edge, where $\xi \gtrsim O(\sqrt{Dt}) \gg 1$, a different expansion is needed, as the transient profile ϕ falls off faster than ϕ^*, so that $\eta \approx -\phi^*$. Linearizing about $\phi = 0$, matching to the interior (6) and imposing that the asymptotic shape ϕ^* is approached for $t \to \infty$ and that the transients are steeper than $e^{-\Lambda\xi}$ for $\xi \to \infty$, uniquely determines the velocity correction $\dot{X}$ (4) and the intermediate asymptotics

$$\phi(\xi, t) \approx e^{-\Lambda\xi - \xi^2/4Dt}[\xi + \text{const} + O(1/\sqrt{t})]. \quad (7)$$

Both the leading $1/t$ term in $\dot{X}(t)$ in (4) and the crossover to a Gaussian type profile like in (7) can be understood intuitively through a heuristic argument [13]: We work in the asymptotic frame $\xi^* = x - v^* t$. Generally, the asymptotic profile is $\phi^*(\xi^*) \propto e^{-\Lambda\xi^*}(\xi^* + \text{const})$ for $\xi^* \to \infty$. The term linear in ξ^* comes from the coincidence of two roots of $\omega(k) - vk$ at a saddle point (3). If we start from localized initial conditions, $\phi(\xi^*, t)$ should approach $\phi^*(\xi^*)$ as $t \to \infty$, but for a fixed time, ϕ should fall off faster than ϕ^* as $\xi^* \to \infty$. To study this crossover, consider for simplicity Eq. (1); if we linearize, and substitute $\phi(\xi^*, t) = e^{-\Lambda\xi^*}\psi(\xi^*, t)$ (with $v^* = 2$, $\Lambda = 1 = D$ in this case), we get the simple diffusion equation $\partial_t\psi = \partial^2_{\xi^*}\psi$. Clearly, the similarity solution which matches to $\phi^*(\xi^*) \sim e^{-\Lambda\xi^*}(\xi^* + \text{const})$ is $\psi \sim (\xi^*/t^{3/2})e^{-\xi^{*2}/4t}$, so $\phi \sim e^{(-\Lambda\xi^* - 3/2\ln t + \ln\xi^* - \xi^{*2}/4t)}$ [22]. Hence, if we now track the position ξ^*_h of the point where $\phi(\xi^*_h, t) = h \ll 1$, we find $\xi^*_h(t) = -3/(2\Lambda)\ln t + \ldots$ in the frame ξ^*. This is precisely the leading term of $X(t)$. We also find here a Gaussian type profile in the far edge, but the systematic analysis sketched below is needed to confirm that (7) is the proper asymptotics in the shifted frame ξ.

We have tested our predictions by numerically integrating Eqs. (1) and (2) forward in time, starting from localized initial conditions. In Figs. 1(a) and 1(b), we present velocities $v_h(t)$ of various points where $\phi(x_h, t) = h$, in 1(a) for Eq. (1), and in 1(b) for Eq. (2) with $\gamma = 0.08$. Note that the critical value of γ is $\gamma_c = 1/12 = 0.083$. As according to our prediction in Eq. (4), $v_h(t) = v^* + \dot{X}(t) + g(h)/t^2$ [where $g(h)$ can be expressed in terms of $\eta_{\rm sh}$ and $\partial_\xi\phi^*$ [17]], we plot $v_h(t) - v^* - \dot{X}(t)$ versus $1/t^2$ for various h. All curves should then converge linearly to zero as $1/t^2 \to 0$. Clearly, the numerical simulations fully confirm this for both equations.

For our prediction (6) of the shape relaxation, the most direct test is to plot $[\phi(\xi, t) - \phi^*(\xi)]/[\dot{X}(t)\eta_{\rm sh}(\xi)]$ as a function of ξ for various times. This ratio should converge to one for large times. As Fig. 1(c) shows, this is fully borne out by our simulations of the nonlinear diffusion equation (1). Moreover, the crossover for large positive ξ is fully in accord with our result that the proper similarity variable in the far edge is ξ^2/t—see Eq. (7).

We finally give a brief sketch of the systematic analysis, taking the nonlinear diffusion equation (1) as an example. Full details will be published elsewhere [17].

We first consider the "front interior" region, where the deviation $\eta(\xi, t)$ of ϕ about $\phi^*(\xi)$ is small, i.e., $|\eta| \ll \phi^*$. As there is some freedom in choosing ξ due to translation invariance, we choose quite arbitrarily the condition that $\phi(0, t) = \frac{1}{2} = \phi^*(0)$, so that $\eta(0, t) = 0$, as was also done in Fig. 1(c). Substituting (5) into (1), we obtain

$$\partial_t\eta = \mathcal{L}^*\eta + \dot{X}\partial_\xi(\eta + \phi^*) + \frac{f''(\phi^*)}{2}\eta^2 + O(\eta^3), \quad (8)$$

$$\mathcal{L}^* = \partial^2_\xi + v^*\partial_\xi + f'(\phi^*(\xi)). \quad (9)$$

The inhomogeneity $\dot{X}\partial_\xi \phi^*$ in (8) is due to the fact that $\phi^*(\xi)$ is a solution of (1) only if $\dot{X} = 0$. Since $\dot{X}(t) = O(t^{-1})$, and since in the front interior $|\eta| \ll \phi^*$, the inhomogeneity induces an ordering in powers of $1/t$, which suggests an asymptotic expansion as

$$\dot{X} = \frac{c_1}{t} + \frac{c_{3/2}}{t^{3/2}} + \frac{c_2}{t^2} + \ldots, \qquad (10)$$

$$\eta(\xi, t) = \frac{\eta_1}{t} + \frac{\eta_{3/2}}{t^{3/2}} + \ldots. \qquad (11)$$

The necessity for actually expanding in powers of $1/\sqrt{t}$ emerges from matching to the similarity solutions in the far edge. Substitution of the above expansions in (8) yields a hierarchy of ordinary differential equations of second order

$$\mathcal{L}^*\eta_1 = -c_1\partial_\xi\phi^*, \quad \mathcal{L}^*\eta_{3/2} = -c_{3/2}\partial_\xi\phi^*,$$
$$\mathcal{L}^*\eta_2 = -c_2\partial_\xi\phi^* - c_1\partial_\xi\eta_1 - \eta_1 - f''(\phi^*)\eta_1^2/2, \qquad (12)$$

etc. The hierarchy is such that the equations can be solved order by order. Each η_i is uniquely determined by its differential equation, the appropriate boundary conditions and the requirement $\eta_i(0) = 0$. The equations for η_1/c_1 and $\eta_{3/2}/c_{3/2}$ are precisely the differential equation for the "shape mode" $\eta_{\text{sh}} = \delta\phi_v/\delta v|_{v^*}$ of (6).

By expanding the η_i for large ξ, one finds that they all behave like $e^{-\Lambda\xi} = e^{-\xi}$ times a polynomial in ξ, whose degree grows with i. The η_i expansion is therefore not properly ordered for large ξ. This just reflects the fact that on the far right, η and ϕ^* must almost cancel each other. This is required for fronts that emerge from localized initial conditions, whose total profile thus decays faster than ϕ^*. A detailed investigation of this region shows that $z = \xi^2/4t$ is a proper similarity variable here, and suggests that here the proper expansion is

$$\phi(\xi, t) = e^{-\xi - z}\left[\sqrt{t}\, g_{\frac{-1}{2}}(z) + g_0(z) + \frac{g_{\frac{1}{2}}(z)}{\sqrt{t}} + \ldots\right]. \qquad (13)$$

Upon substitution of this expansion into the original partial differential equation, linearized about $\phi = 0$, we now find a different hierarchy of ordinary differential equations for the functions $g_{n/2}(z)$. In this case, the conditions to be imposed on the $g_{n/2}$'s is that they do not diverge as e^z as $z \to \infty$, and that they match, in the language of matched asymptotic expansions, the large ξ "outer" expansion of the "inner" solution based on the η_i [23]. These conditions fix the parameters c_1 and $c_{3/2}$ in (10), and this yields the solution given in Eqs. (4)–(7) [17]. The structure of the analysis is essentially the same for higher order equations like (2).

In summary, our results show that the $1/t$ relaxation of pulled fronts is essentially due to the crossover to a Gaussian shaped tip in the leading edge of the front. The nonlinearities dictate the asymptotic tip shape $\phi^* \propto \xi e^{-\Lambda\xi}$ for $t \to \infty$ and ξ large. This asymptote determines the coefficients and the $1/t^{3/2}$ term in the velocity correction $\dot{X}$ (4). We finally note that analytical arguments as well as numerical simulations indicate that many of the above arguments can be generalized to pattern forming fronts, occurring, e.g., in Eq. (2) for $\gamma > 1/12$ or in the Swift-Hohenberg equation [13]. Work on this is in progress.

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