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Exercise Set 2

Exercise 2.1 ☺ Two orthonormal bases $\{|e_i\rangle\}_{i\in I}$ and $\{|f_j\rangle\}_{j\in J}$ of a d-dimensional Hilbert space $\mathcal{H}$ are called **mutually unbiased** if

$$\big|\langle e_i|f_j\rangle\big|^2 = \frac{1}{d}$$

for all $i \in I$ and $j \in J$. For instance, we can see that the computational basis and the Hadamard basis are mutually unbiased. Find a third orthonormal basis of $\mathbb{C}^2$ so that out of the three ($\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$ and the new one) *any two* are mutually unbiased.

Exercise 2.2 ☺ For $|\varphi\rangle, |\psi\rangle \in \mathcal{S}(\mathcal{H})$, the **fidelity** is given by $F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|$ (see Def. 1.7). On the other hand, for two probability distributions $p = \{p_i\}_{i\in I}$ and $q = \{q_i\}_{i\in I}$, the fidelity (or **Bhattacharyya coefficient**) is defnied as $F(p,q) := \sum_i \sqrt{p_i q_i}$. Show that for any two state vectors $|\varphi\rangle, |\psi\rangle \in \mathcal{S}(\mathcal{H})$ and for any orthonormal basis $\{|e_i\rangle\}_{i\in I}$ of $\mathcal{H}$, the probability distributions p and q given by $p_i = |\langle e_i|\varphi\rangle|^2$ and $q_i = |\langle e_i|\psi\rangle|^2$ are such that $F(p,q) \geq F(|\varphi\rangle, |\psi\rangle)$. Show the same for the general case where $p_i = \|M_i|\varphi\rangle\|^2$ and $q_i = \|M_i|\psi\rangle\|^2$ with $\{M_i\}_{i\in I}$ an arbitrary measurement.

Exercise 2.3 ☺ How does the Hadamard operator $H \in \mathcal{L}(\mathbb{C}^2)$ act as a map on the Bloch sphere? Formally, if $\rho = \frac{1}{2}(\mathbb{I} + xX + yY + zZ)$ for $(x,y,z) \in \mathbb{R}^3$, what are the "Bloch-sphere coordinates" $(x',y',z') \in \mathbb{R}^3$ that satisfy $H\rho H^\dagger = \frac{1}{2}(\mathbb{I} + x'X + y'Y + z'Z)$? Do you see, and can you explain in words, what the map $(x,y,z) \mapsto (x',y',z')$ on the bloch sphere does?

Exercise 2.4 ☺ Show that the unitary $R_X(\theta) = \cos\big(\frac{\theta}{2}\big)\mathbb{I} - i\sin\big(\frac{\theta}{2}\big)X \in \mathcal{U}(\mathbb{C}^2)$ satisfies

$$R_X(\theta)R_X(\theta') = R_X(\theta+\theta')$$

for all $\theta, \theta' \in \mathbb{R}$. This supports our understanding of the unitary $R_X(\theta)$ being a rotation (of the Bloch sphere) with angle θ.

Exercise 2.5 ☺ For $\mathcal{H}_1 = \mathbb{C}^2 = \mathcal{H}_2$ and $|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as specified below, determine whether $|\Phi\rangle$ is a pure tensor, i.e., $|\Phi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$ for $|\varphi_1\rangle \in \mathcal{H}_1$, $|\varphi_2\rangle \in \mathcal{H}_2$. In case it is, provide a tensor decomposition; otherwise, you may claim it without showing it.

1. $|\Phi\rangle = |0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle$.
2. $|\Phi\rangle = |0\rangle \otimes |-\rangle - |1\rangle \otimes |+\rangle$.
3. $|\Phi\rangle = |1\rangle \otimes |0\rangle + |0\rangle \otimes |+\rangle + |1\rangle \otimes |1\rangle$.
4. $|\Phi\rangle = i|0\rangle|0\rangle + 2|0\rangle|1\rangle + |1\rangle|0\rangle + 2i|1\rangle|1\rangle$.

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Exercise 2.6 Let $|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a non-zero vector, and let $\{|e_i\rangle\}_{i\in I}$ be an ONB of $\mathcal{H}_1$. Consider the operator

$$A := \sum_{i\in I} \big(\langle e_i| \otimes \mathbb{I}_2\big)|\Phi\rangle\langle e_i| \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)\,,$$

where $\mathbb{I}_2$ is the identity on $\mathcal{H}_2$. First, verify that

$$|\Phi\rangle = \sum_{i\in I} |e_i\rangle \otimes A|e_i\rangle\,.$$

Hint: By linearity, it is sufficient to show the equality for the case where $|\Phi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$.

Second, show that $|\Phi\rangle$ is a pure tensor if and only if $\mathrm{rank}(A) = 1$.
Hint: Use the fact that $\mathrm{rank}(A) = 1 \iff A = |\psi_2\rangle\langle\psi_1|$ for some $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$.

Finally, for the case(s) in Exercise 2.5 for which you were not able to write $|\Phi\rangle$ as a pure tensor, verify if $|\Phi\rangle$ is indeed not a pure tensor by the above means.

Remark: More general, the rank of A coincides with the minimum number of pure tensors that linearly combine to $|\Phi\rangle$. This quantity is called the (bipartite) tensor rank of $|\Phi\rangle$.