Solutions to Exercise Set 2

Solution 2.1 It is not hard to guess and then easy to verify that $|\circlearrowleft\rangle$ and $|\circlearrowright\rangle$ do the job. For a formal derivation, which also shows uniqueness (up to global phases), see the following.

In order to be mutually unbiased with $\{|0\rangle, |1\rangle\}$, such a basis needs to be of the form

$$|e_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\omega}{\sqrt{2}}|1\rangle \qquad \text{and} \qquad |e_1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{\omega}{\sqrt{2}}|1\rangle$$

for some $\omega \in \mathcal{S}(\mathbb{C})$, up to individual global phases. Asking for

$$\frac{1}{2} \stackrel{!}{=} |\langle \pm|e_0\rangle|^2 = \left|\frac{1}{2} \pm \frac{\omega}{2}\right|^2 = \left(\frac{1}{2} \pm \frac{\omega}{2}\right)\left(\frac{1}{2} \pm \frac{\bar{\omega}}{2}\right) = \frac{1}{2} \pm \frac{\omega + \bar{\omega}}{2} = \frac{1}{2} \pm \Re(\omega)$$

then enforces the real part of ω to be 0; thus, $\omega = \pm i$ are the only options. Finally, given that $\langle \pm|e_1\rangle = \frac{1}{2} \mp \frac{\omega}{2}$, it is then easy to see that this choice indeed works.

Note that this basis is the eigenbasis of Y.

Solution 2.2 We simply have

$$F(p,q) = \sum_i \sqrt{p_i q_i} = \sum_i \sqrt{|\langle e_i|\varphi\rangle|^2 |\langle e_i|\psi\rangle|^2} = \sum_i \big|\langle \varphi|e_i\rangle\langle e_i|\psi\rangle\big| \geq \left|\sum_i \langle \varphi||e_i\rangle\langle e_i||\psi\rangle\right| = |\langle \varphi|\psi\rangle|,$$

where the inequality is triangle inequality. The general case is argued similarly:

$$F(p,q) = \sum_i \|M_i|\varphi\rangle\| \|M_i|\psi\rangle\| \geq \sum_i \big|\langle \varphi|M_i^\dagger M_i|\psi\rangle\big| \geq \left|\sum_i \langle \varphi|M_i^\dagger M_i|\psi\rangle\right| = |\langle \varphi|\psi\rangle|\,,$$

except that Cauchy-Schwarz inequality (Proposition 0.1) is used as well.

Solution 2.3 Using the results from Exercise 1.3, we obtain that

$$H\rho H^\dagger = \frac{1}{2}(\mathbb{I} + xHXH + yHYH + zHZH) = \frac{1}{2}(\mathbb{I} + zX - yY + xZ)\,.$$

Thus, $(x,y,z) \mapsto (x',y',z') = (z,-y,x)$, which is a rotation by 180° around the diagonal axis in-between the x- and the z-axis, i.e., around the axis that is defined by the point on the Bloch sphere given by the (appropriately normalized) vector $|0\rangle + |+\rangle$.

Solution 2.5 For 1., we have

$$|\Phi\rangle = |0\rangle \otimes (|0\rangle + |1\rangle) + |1\rangle \otimes (|0\rangle + |1\rangle) = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\,.$$

For 2., it turns out that $|\Phi\rangle$ cannot be written as a pure tensor. For 3., we have

$$|\Phi\rangle = |0\rangle \otimes |+\rangle + |1\rangle \otimes (|0\rangle + |1\rangle) = |0\rangle \otimes |+\rangle + |1\rangle \otimes \sqrt{2}|+\rangle = \big(|0\rangle + \sqrt{2}|1\rangle\big) \otimes |+\rangle\,.$$

Finally, for 4., $|\Phi\rangle$ cannot be written as a pure tensor.

Solution 2.6 For $|\Phi\rangle = |\varphi_1\rangle|\varphi_2\rangle$, the definition of A simplifies to

$$A := \sum_{i\in I} \big(\langle e_i| \otimes \mathbb{I}_2\big)\big(|\varphi_1\rangle \otimes |\varphi_2\rangle\big)\langle e_i| = \sum_{i\in I} \langle e_i|\varphi_1\rangle \otimes |\varphi_2\rangle\langle e_i| = \sum_{i\in I} \langle e_i|\varphi_1\rangle|\varphi_2\rangle\langle e_i|$$

and so

$$\begin{aligned}\sum_{i\in I} |e_i\rangle \otimes A|e_i\rangle &= \sum_{i\in I} |e_i\rangle \otimes \sum_{j\in I} \langle e_j|\varphi_1\rangle|\varphi_2\rangle\langle e_j|e_i\rangle = \sum_{i\in I} |e_i\rangle \otimes \langle e_i|\varphi_1\rangle|\varphi_2\rangle \\ &= \sum_{i\in I} |e_i\rangle\langle e_i|\varphi_1\rangle \otimes |\varphi_2\rangle = \sum_{i\in I} |e_i\rangle\langle e_i||\varphi_1\rangle \otimes |\varphi_2\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle = |\Phi\rangle\,.\end{aligned}$$

Towards the second claim, we first not that if $|\Phi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$, i.e., is not entangled, then from further rewriting the above we see that

$$A = \sum_{i\in I} \langle e_i|\varphi_1\rangle|\varphi_2\rangle\langle e_i| = \sum_{i\in I} |\varphi_2\rangle\langle e_i|\varphi_1\rangle\langle e_i| = |\varphi_2\rangle\Big(\sum_{i\in I} \langle e_i|\varphi_1\rangle\langle e_i|\Big)\,,$$

and thus A has rank 1. On the other hand, if A has rank 1, i.e., if $A = |\psi_2\rangle\langle\psi_1|$, then

$$|\Phi\rangle = \sum_{i\in I} |e_i\rangle \otimes A|e_i\rangle = \sum_{i\in I} |e_i\rangle \otimes |\psi_2\rangle\langle\psi_1|e_i\rangle = \Big(\sum_{i\in I} |e_i\rangle\langle\psi_1|e_i\rangle\Big) \otimes |\psi_2\rangle$$

and thus is a product state, i.e., not entangled.

Finally, for $|\Phi\rangle = |0\rangle \otimes |-\rangle - |1\rangle \otimes |+\rangle$ in Exercise 2.5, taking the computational basis we obtain

$$\begin{aligned}A &= \big(\langle 0| \otimes \mathbb{I}_2\big)|\Phi\rangle\langle 0| + \big(\langle 1| \otimes \mathbb{I}_2\big)|\Phi\rangle\langle 1| = |-\rangle\langle 0| - |+\rangle\langle 1| \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\,,\end{aligned}$$

which indeed has rank 2. Similarly for $|\Phi\rangle = i|0\rangle|0\rangle + 2|0\rangle|1\rangle + |1\rangle|0\rangle + 2i|1\rangle|1\rangle$, we we obtain

$$A = i|0\rangle\langle 0| + 2|1\rangle\langle 0| + |0\rangle\langle 1| + 2i|i\rangle\langle i| = \begin{bmatrix} i & 1 \\ 2 & 2i \end{bmatrix}\,,$$

which has rank 2.