Appendix B

(Operator) Monotonicity & Convexity

B.1 Operator Monotonicity

Definition B.1. *A function $f : D \to \mathbb{R}$ on a subset $D \subseteq \mathbb{R}$ is called* **monotone** *if*

$$x \leq y \implies f(x) \leq f(y)$$

for all $x, y \in D$. f is called **anti-monotone** *if $-f$ is monotone, i.e., if $x \leq y \Rightarrow f(x) \geq f(y)$.*

We are particularly interested in the following extensions of the notion of a (anti-)monotone function to *operator functions*, as originally considered and studied by Loewner.

Remark B.1. We may replace the domain and/or the range of the function f by (a subset of) $\mathcal{L}(\mathcal{H})$ and understand $\leq$ in terms of the Loewner order then. This way, we obtain a notion of *(anti-)monotonicity* for functions that act on *operators* and/or that are *operator-valued.* For example, for a function $f : D \to \mathbb{R}$ as in Definition B.1, we may consider it as a function acting on Hermitian operators $X \in \mathcal{L}(\mathcal{H})$ (with eigenvalues in D) by means of Definition 0.1, and we may then ask whether f is (anti-)monotone as such an operator function. If this is the case *for any choice of* $\mathcal{H}$ then we say that f is **operator (anti-)monotone**.

Proposition B.1. *The function $(0,\infty) \to (0,\infty)$, $x \mapsto \frac{1}{x}$ is operator anti-monotone.*

Proof. Let $R \in \mathcal{P}(\mathcal{H})$ be invertivle and $L = R + \Delta$ for $\Delta \in \mathcal{P}(\mathcal{H})$. We show that $\frac{1}{R} - \frac{1}{L} \geq 0$. It is easy to verify that

$$\frac{1}{R} - \frac{1}{R+\Delta} = \frac{1}{\sqrt{R}}\Big(\mathbb{I} - \sqrt{R}\,\frac{1}{R+\Delta}\sqrt{R}\Big)\frac{1}{\sqrt{R}} = \frac{1}{\sqrt{R}}\left(\mathbb{I} - \frac{1}{\mathbb{I} + \frac{1}{\sqrt{R}}\Delta\frac{1}{\sqrt{R}}}\right)\frac{1}{\sqrt{R}}\,.$$

Furthermore, for any $K \in \mathcal{L}(\mathcal{H})$, if $K \geq 0$ then $\mathbb{I} - \frac{1}{\mathbb{I}+K} \geq 0$ as well. This proves the claim. □

As a direct consequence, we see that also $(0,\infty) \to (0,\infty)$, $x \mapsto \frac{1}{t+x}$ is operator monotone for any $t \geq 0$, while $(0,\infty) \to (0,\infty)$, $x \mapsto \frac{x}{t+x} = 1 - \frac{1}{t+x}$ is operator anti-monotone.

The operator anti-monotonicity of $\frac{1}{x}$ is obviously a special case of the following general result on the operator (anti-)monotonicity of the function x^s.

Theorem B.2. *The function $(0,\infty) \to (0,\infty)$, $x \mapsto x^s$ is operator monotone for $0 \leq s \leq 1$ and operator anti-monotone for $-1 \leq s \leq 0$.*

It is known that for any exponent s outside of the two above ranges, the function x^s is *not* operator (anti-)monotone. On the positive side, $\log(x)$ is another operator monotone function. For the proof, as well as for later purposes, we need the following technical lemma.

Lemma B.3. *For any* $0 < s < 1$ *there exists a constant* $\kappa > 0$ *so that for any* $x > 0$

$$x^{s-1} = \kappa \int_0^\infty \frac{t^{s-1}}{t+x} dt \qquad \text{and} \qquad x^s = \kappa \int_0^\infty \frac{x}{t+x} t^{s-1} dt \,.$$

Using contour integration, one obtains $\kappa = \frac{\pi}{\sin(\pi s)}$, but this is not important for us.

Proof. We first point out that the integrals exists. Indeed, the integrand is bounded by t^{s-1}/x with $s-1 > -1$, and so the integral exists in the neighborhood of 0, and the integrand is bounded as well by t^{s-2} with $s-2 < -1$, and so the integral exists in the neighborhood of ∞. Furthermore, by the variable substitution $t = xu$, we see that

$$\int_0^\infty \frac{t^{s-1}}{t+x} dt = x^{s-1} \int_0^\infty \frac{u^{s-1}}{u+1} du \,,$$

and thus the first claim follows by letting κ be the inverse of the integral on the right hand side. The second claim follows by multiplying with x. □

Corollary B.4. *For any* $0 < s < 1$ *there exists a constant* $\kappa > 0$ *so that for any* $X \in \mathcal{P}^\star(\mathcal{H})$

$$X^{s-1} = \kappa \int_0^\infty \frac{t^{s-1}}{t\mathbb{I} + X} dt \qquad \text{and} \qquad X^s = \kappa \int_0^\infty \frac{X}{t\mathbb{I} + X} t^{s-1} dt \,.$$

Proof. Write X in spectral decomposition, $X = \sum_i x_i |i\rangle\langle i|$, and observe that

$$X^{s-1} = \sum_i x_i^{s-1} |i\rangle\langle i| = \sum_i \kappa \int_0^\infty \frac{t^{s-1}}{t+x} dt\, |i\rangle\langle i| = \kappa \int_0^\infty \sum_i \frac{t^{s-1}}{t+x} |i\rangle\langle i| dt = \kappa \int_0^\infty \frac{t^{s-1}}{t\mathbb{I} + X} dt \,,$$

where we used linearity of the integral. □

Proof of Theorem B.2. Using Proposition B.1 and basic observations, we have for any $t \geq 0$:

$$X \leq Y \implies t\mathbb{I} + X \leq t\mathbb{I} + Y \implies \frac{t^{s-1}}{t\mathbb{I} + X} \geq \frac{t^{s-1}}{t\mathbb{I} + Y} \implies \int_0^\infty \frac{t^{s-1}}{t\mathbb{I} + X} dt \geq \int_0^\infty \frac{t^{s-1}}{t\mathbb{I} + Y} dt \,.$$

Using the above integral representation of X^s and Y^s, we obtain the claim on x^s for $-1 \leq s \leq 0$. The case $0 \leq s \leq 1$ works similarly but using the second integral representation and the operator monotonicity of $\frac{x}{t+x}$ instead. □

B.2 Convexity and Concavity

Operator monotonicity is strongly related to operator convexity, as introduced below. We do not make this relation explicit here, but treat to two as two separate interesting concepts.

Definition B.2. *A set* C*, subset of a real vector space, is called* **convex** *if* $px + (1-p)x' \in C$ *for any* $0 \leq p \leq 1$ *and any* $x, x' \in C$.

Definition B.3. *A function* $f : C \to \mathbb{R} \cup \{\infty\}$ *on a convex set* C *is called* **convex** *if*

$$pf(x) + (1-p)f(x') \geq f\big(px + (1-p)x'\big)$$

for any $0 \leq p \leq 1$ *and any* $x, x' \in C$*. If* $g := -f$ *is convex then* f *is called* **concave**.

We take it as understood that the exponentiation function $\mathbb{R} \to (0,\infty)$, $x \mapsto 2^x$ (and for any other fixed basis > 1) is convex, and the logarithm $\log : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ is concave. Furthermore, $[0,\infty) \to [0,\infty)$, $x \mapsto x^s$ is convex for $s \geq 1$ or $s < 0$, and concave for $0 < s \leq 1$.

The following is obtained by means of a straightforward induction argument.

Proposition B.5 (Jensen's inequality)**.** *Let $f : C \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then, for any $x_1, \ldots, x_n \in C$ and any $0 \leq p_1, \ldots, p_n \in \mathbb{R}$ with $\sum_i p_i = 1$,*

$$\sum_i p_i f(x_i) \geq f\Big(\sum_i p_i x_i\Big) .$$

Remark B.2. If f is a function of several variables, like $f(x,y)$, and f is convex (or concave) when the vector of variables is treated as a single argument, then one also refers to f as being **jointly convex** (respectively **jointly concave**). We emphasize that this is different from being convex in x and in y, i.e., as functions $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$. The following is not too hard to show. If $x \mapsto f(x,y)$ is convex for every y then $x \mapsto \max_y f(x,y)$ is convex, and if f is *jointly* convex then $x \mapsto \min_y f(x,y)$ is convex.

The following theorem is stated here without a proof.

Theorem B.6 (Von Neumann's Minimax Theorem)**.** *Let C and D be compact convex sets, and let $f : C \times D \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ be a continuous function that is convex in y (for any $x \in C$) and concave in x (for any $y \in D$). Then*

$$\min_{y \in D} \max_{x \in C} f(x,y) = \max_{x \in C} \min_{y \in D} f(x,y) .$$

Remark B.3. In line with Remark B.1, we may replace the range of the function f in Defintion B.3 by $\mathcal{L}(\mathcal{H})$ and understand $\geq$ in terms of the Loewner order. This way, we obtain a notion of *convexity* (and *concavity*) for *operator-valued* functions. For example, for a function $f : C \to \mathbb{R} \cup \{\infty\}$ as in Definition B.3, we may consider it as a function acting on Hermitian operators $X \in \mathcal{L}(\mathcal{H})$ (with eigenvalues in C) by means of Definition 0.1, and we may then ask whether f is convex as such an operator-valued function. If this is the case *for any choice of* $\mathcal{H}$ then we say that f is **operator-convex**.

B.3 Operator Jensen Inequality

For operator-convex function, we have the following operator-version of Jensen's inequality.

Theorem B.7 (Jensen's operator inequality)**.** *Let $f : C \to \mathbb{R} \cup \{\infty\}$ be an operator-convex function. Then, for any Hilbert space $\mathcal{H}$ and any Hermitian $X_1, \ldots, X_n \in \mathcal{L}(\mathcal{H})$ with eigenvalues in C, and for any $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{H})$ with $\sum_i A_i^\dagger A_i = \mathbb{I}$,*

$$\sum_i A_i^\dagger f(X_i) A_i \geq f\Big(\sum_i A_i^\dagger X_i A_i\Big) .$$

For the proof, we first show the following. In the remainder, it is understood that f is as above.

Lemma B.8. *Let $\mathcal{H}^\circ, \mathcal{H}$ be arbitrary Hilbert spaces. Then, for any Hermitian $X \in \mathcal{L}(\mathcal{H}^\circ \otimes \mathcal{H})$ with eigenvalues in C, and any basis $\{|0\rangle, \ldots, |n-1\rangle\}$ of $\mathcal{H}^\circ$, it holds that*

$$\sum_k |k\rangle\langle k| \otimes f(X_k) \leq \sum_k |k\rangle\langle k| \otimes f(X)_k$$

where $X_k = (\langle k| \otimes \mathbb{I}) X (|k\rangle \otimes \mathbb{I})$ and similarly $f(X)_k$. In particular, $f(X_k) \leq f(X)_k$ for any k.

Proof. It is straightforward to verify that $\Omega := \sum_k \omega_n^k |k\rangle\langle k|$ with $\omega_n := e^{-2\pi i/n}$ is in $\mathcal{U}(\mathbb{C}^n)$ and

$$\frac{1}{n}\sum_k \Omega^{-k} Y \Omega^k = \sum_k |k\rangle\langle k|Y|k\rangle\langle k| = \sum_k |k\rangle\langle k| \cdot \langle k|Y|k\rangle ,$$

for any $Y \in \mathcal{L}(\mathbb{C}^n)$, and thus

$$\frac{1}{n}\sum_k (\Omega^{-k} \otimes \mathbb{I}) X (\Omega^k \otimes \mathbb{I}) = \sum_k |k\rangle\langle k| \otimes (\langle k| \otimes \mathbb{I}) X (|k\rangle \otimes \mathbb{I})$$

for any $X \in \mathcal{L}(\mathcal{H}^\circ \otimes \mathcal{H})$. Using Ω as a short hand for $\Omega \otimes \mathbb{I}$, we obtain that

$$\begin{aligned}\sum_k |k\rangle\langle k| \otimes f(X_k) &= f\Big(\sum_k |k\rangle\langle k| \otimes X_k\Big) = f\Big(\frac{1}{n}\sum_k \Omega^{-k} X \Omega^k\Big) \\ &\leq \frac{1}{n}\sum_k f(\Omega^{-k} X \Omega^k) = \frac{1}{n}\sum_k \Omega^{-k} f(X) \Omega^k = \sum_k |k\rangle\langle k| \otimes f(X)_k ,\end{aligned}$$

as claimed. □

Using that every isometry $V \in \mathcal{L}(\mathcal{H}, \mathcal{H}^\circ \otimes \mathcal{H})$ can be written as $V = U(|0\rangle \otimes \mathbb{I})$ for a unitary operator $U \in \mathcal{U}(\mathcal{H}^\circ \otimes \mathcal{H})$, we get the following.

Corollary B.9. *For X as above and $V \in \mathcal{L}(\mathcal{H}, \mathcal{H}^\circ \otimes \mathcal{H})$ an isometry: $f(V^\dagger X V) \leq V^\dagger f(X) V$.*

Proof of Theorem B.7. We apply Corollary B.9 to

$$X := \sum_i |i\rangle\langle i| \otimes X_i \in \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}) \qquad \text{and} \qquad V = \sum_i |i\rangle \otimes A_i .$$

Observing that $V^\dagger X V = \sum_i A_i^\dagger X_i A_i$ and $V^\dagger f(X) V = \sum_i A_i^\dagger f(X_i) A_i$ then proves the claim. □

B.4 Some Important (Operator-)Convexity/Concavity Results

Below, we write $\mathcal{P}^\star(\mathcal{H})$ to denote the set of *invertible* operators in $\mathcal{P}(\mathcal{H})$. We start with the following technical observation, which follows immediately from the fact that

$$\begin{bmatrix} \mathbb{I} & -XR^{-1} \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} Y & X \\ X^\dagger & R \end{bmatrix} \begin{bmatrix} \mathbb{I} & 0 \\ -R^{-1}X^\dagger & \mathbb{I} \end{bmatrix} = \begin{bmatrix} Y - XR^{-1}X^\dagger & 0 \\ 0 & R \end{bmatrix} ,$$

and that $\begin{bmatrix} \mathbb{I} & -XR^{-1} \\ 0 & \mathbb{I} \end{bmatrix}$ is invertible, its inverse being $\begin{bmatrix} \mathbb{I} & XR^{-1} \\ 0 & \mathbb{I} \end{bmatrix}$.

Lemma B.10. *For arbitrary $X, Y \in \mathcal{L}(\mathcal{H})$ and invertible $R \in \mathcal{P}(\mathcal{H})$:*

$$\begin{bmatrix} Y & X \\ X^\dagger & R \end{bmatrix} \geq 0 \iff Y \geq XR^{-1}X^\dagger$$

Proposition B.11. *The map*

$$\mathcal{L}(\mathcal{H}) \times \mathcal{P}^\star(\mathcal{H}) \to \mathcal{L}(\mathcal{H}),\ (L, R) \mapsto L(R^{-1}) := LR^{-1}L^\dagger$$

is jointly convex, and the maps

$$\mathcal{P}(\mathcal{H}) \times \mathcal{P}^\star(\mathcal{H}) \to \mathcal{L}(\mathcal{H}),\ (L, R) \mapsto L\#R := R^{1/2}\big(R^{-1/2} L R^{-1/2}\big)^{1/2} R^{1/2} ,$$

and

$$\mathcal{P}^\star(\mathcal{H}) \times \mathcal{P}^\star(\mathcal{H}) \to \mathcal{L}(\mathcal{H}),\ (L, R) \mapsto L!R := \frac{1}{\frac{1}{L} + \frac{1}{R}} ,$$

respectively referred to as the **geometric** *and the* **harmonic mean***, are jointly concave.*

The first statement in particular implies that $L \mapsto L^2$ and $R \mapsto R^{-1}$ are convex, and the second statement implies that $L \mapsto \sqrt{L}$ is concave.

Proof. Let $L_1, L_2 \in \mathcal{L}(\mathcal{H})$ and $R_1, R_2 \in \mathcal{P}^\star(\mathcal{H})$, and $0 \le p \le 1$. By Lemma B.10,

$$
\begin{aligned}
0 &\le p \begin{bmatrix} L_1(R_1^{-1}) & L_1 \\ L_1^\dagger & R_1 \end{bmatrix} + (1-p) \begin{bmatrix} L_2(R_2^{-1}) & L_2 \\ L_2^\dagger & R_2 \end{bmatrix} \\
&= \begin{bmatrix} pL_1(R_1^{-1}) + (1-p)L_2(R_2^{-1}) & pL_1 + (1-p)L_2 \\ pL_1^\dagger + (1-p)L_2^\dagger & pR_1 + (1-p)R_2 \end{bmatrix}
\end{aligned}
$$

and thus the first claim follows by invoking Lemma B.10 once more.

Exploiting that $(L\#R)R^{-1}(L\#R) = L$ and using Lemma B.10, we see that

$$
\begin{aligned}
0 &\le p \begin{bmatrix} L_1 & L_1\#R_1 \\ L_1\#R_1 & R_1 \end{bmatrix} + (1-p) \begin{bmatrix} L_2 & L_2\#R_2 \\ L_2\#R_2 & R_2 \end{bmatrix} \\
&= \begin{bmatrix} pL_1 + (1-p)L_2 & pL_1\#R_1 + (1-p)L_2\#R_2 \\ pL_1\#R_1 + (1-p)L_2\#R_2 & pR_1 + (1-p)R_2 \end{bmatrix}
\end{aligned}
$$

To conclude, we observe that if $\left[\begin{smallmatrix} L & X \\ X & R \end{smallmatrix}\right] \ge 0$ for $X \ge 0$ then, by Lemma B.10 again, $L \ge XR^{-1}X$, and thus

$$
R^{-1/2}LR^{-1/2} \ge R^{-1/2}XR^{-1}XR^{-1/2} = \left(R^{-1/2}XR^{-1/2}\right)^2 .
$$

Therefore, by the operator monotonicity of the square root (Theorem B.2), we have the relation $\left(R^{-1/2}LR^{-1/2}\right)^{1/2} \ge R^{-1/2}XR^{-1/2}$, and thus $L\#R \ge X$.

To investigate the harmonic mean, we use

$$
\frac{1}{\frac{1}{L} + \frac{1}{R}} = R - R(L+R)^{-1}R ,
$$

which is (a special case of) the **Woodbury matrix identity** and can be verified by multiplying both sides with $L^{-1} + R^{-1}$ and doing some proper manipulations. To conclude the proof, we note that $R(L+R)^{-1}R$ is jointly convex due to the first statement and the fact that $L + R$ is linear in L and R. □

The following combines **Lieb's Concavity** and **Ando's Convexity Theorem**.

Theorem B.12. *The map*

$$
\mathcal{P}^\star(\mathcal{H}) \times \mathcal{P}^\star(\mathcal{H}') \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}'),\ (L, R) \mapsto L^s \otimes R^{1-s}
$$

is jointly concave for $0 \le s \le 1$ *and jointly convex for* $-1 \le s \le 0$ *or* $1 \le s \le 2$.

Remark B.4. For $0 \le s \le 1$, the restriction to *invertible* operators can be dropped by continuity. For the other case and considering the pseudo-inverse, the restriction can be dropped if the supports of L and R are appropriate.

Proof. Let $s \in (0,1)$. Using the second integral representation from Lemma B.3, we can write

$$
L^s \otimes R^{1-s} = (L \otimes R^{-1})^s \cdot (\mathbb{I} \otimes R) = \kappa \int_0^\infty \frac{L \otimes R^{-1}}{t\mathbb{I} \otimes \mathbb{I} + L \otimes R^{-1}} (\mathbb{I} \otimes R) t^{s-1} dt ,
$$

where we exploit that we can think of L and R to be diagonal, and so everything commutes nicely. Noting that the integrand is $(tL^{-1} \otimes \mathbb{I} + \mathbb{I} \otimes R^{-1})^{-1} = \big(\frac{1}{t} L \otimes \mathbb{I}\big) ! \big(\mathbb{I} \otimes R\big)$, the claim follows from Proposition B.11, given that $\frac{1}{t} L \otimes \mathbb{I}$ and $\mathbb{I} \otimes R$ are both linear in L and R.

Similarly, for $s \in (1, 2)$, we can write

$$L^s \otimes R^{1-s} = (L \otimes R^{-1})^{s-1} \cdot (L \otimes \mathbb{I}) = \kappa \int_0^\infty \frac{L \otimes R^{-1}}{t\mathbb{I} \otimes \mathbb{I} + L \otimes R^{-1}} (L \otimes \mathbb{I}) t^{s-2} dt \,.$$

Noting that the integrand is $(L \otimes \mathbb{I})(t\mathbb{I} \otimes R + L \otimes \mathbb{I})^{-1}(L \otimes \mathbb{I})$, the claim holds by Proposition B.11, given that $L \otimes \mathbb{I}$ and $t\mathbb{I} \otimes R + L \otimes \mathbb{I}$ are both linear in L and R. Finally, the case $s \in (-1, 0)$ follows by symmetry, and the extreme points by continuity. □

From Corollary 0.5 (and the linearity of the transposition map) we obtain the following version of Lieb's Concavity and Ando's Convexity Theorem.

Corollary B.13. *For any $K \in \mathcal{L}(\mathcal{H})$, the map*

$$\mathcal{P}^\star(\mathcal{H}) \times \mathcal{P}^\star(\mathcal{H}) \to \mathbb{R}, (L, R) \mapsto \operatorname{tr}\big(L^s K R^{1-s} K^\dagger\big)$$

is jointly concave for $0 \leq s \leq 1$ and jointly convex for $1 \leq s \leq 2$.

As another corollary, we obtain the following. It is known that outside this range for s the function x^s is *not* operator convex/concave.

Corollary B.14. *The function $(0, \infty) \to (0, \infty)$, $x \mapsto x^s$ is operator convex for $1 \leq s \leq 2$ and $-1 \leq s \leq 0$, and operator concave for $0 \leq s \leq 1$.*