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Exercise Set 1

Exercise 1.1 We have seen that $\mathcal{D}(\mathcal{H})$ coincides with the convex hull of the set of pure density operators in $\mathcal{D}(\mathcal{H})$. Show that the pure density operators are the *extreme points* in $\mathcal{D}(\mathcal{H})$, i.e., if

$$|\varphi\rangle\langle\varphi| = \sum_{i=1}^{n} \varepsilon_i |\varphi_i\rangle\langle\varphi_i|$$

with $|\varphi\rangle, |\varphi_1\rangle, \ldots, |\varphi_n\rangle \in \mathcal{S}(\mathcal{H})$, $\varepsilon_1, \ldots, \varepsilon_n > 0$ and $\sum_i \varepsilon_i = 1$, then $|\varphi\rangle\langle\varphi| = |\varphi_i\rangle\langle\varphi_i|$ for every i.
Hint: Apply $\langle\varphi|$ from the left and $|\varphi\rangle$ from the right, and use Proposition 0.1 (Chauchy-Schwarz).

Exercise 1.2 Let $\rho \in \mathcal{D}(\mathcal{H})$. Show that $0 \leq \mathrm{tr}(\rho^2) \leq 1$, with $\mathrm{tr}(\rho^2) = 1$ if and only if ρ is pure.
Hint: Use spectral decomposition.

Exercise 1.3 Show that for any two $\rho, \sigma \in \mathcal{D}(\mathcal{H})$: $0 \leq \mathrm{tr}(\rho\,\sigma) \leq 1$.

Exercise 1.4 Show that two states that are described by *different* respective density matrices $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, can be distinguished with positive advantage by a suitable measurement. Concretely, show that if $\rho \neq \sigma$ then there exists an orthonormal basis $\{|i\rangle\}_{i \in I}$ of $\mathcal{H}$ such that $\mathrm{tr}(|i\rangle\langle i|\rho) \neq \mathrm{tr}(|i\rangle\langle i|\sigma)$ for some $i \in I$.
Hint: Consider the spectral decomposition of $\rho - \sigma$.

Exercise 1.5 Let $A \in \mathcal{L}(\mathcal{H})$. Show that if $\langle\varphi|A|\varphi\rangle = 0$ for all $|\varphi\rangle \in \mathcal{H}$ then $A = 0$.
Hint: Show that $\langle\varphi|A|\varphi\rangle = 0$ for all $|\varphi\rangle \in \mathcal{H}$ implies that $\langle e_i|A|e_j\rangle = 0$ for all $i, j \in I$ and any orthonormal basis $\{|e_i\rangle\}_{i \in I}$ of $\mathcal{H}$. Also, it is helpful to realize that the statement is not true in case $\mathcal{H}$ is a *real* Hilbert space.
NB: You are not meant to use the property that $A \geq 0 \Rightarrow A$ Hermitian; as a matter of fact, what you are supposed to show is typically used to prove this property.