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# Exercise Set 2

**Exercise 2.1** Let $\rho_A \in \mathcal{D}(\mathcal{H}_A)$, and let $\sqrt{\rho_A} \in \mathcal{L}(\mathcal{H}_A)$ be the positive-semidefinite square root of $\rho_A$, i.e., $\sqrt{\rho_A} \geq 0$ and $\sqrt{\rho_A}^2 = \rho_A$.[1] Also, let $\mathcal{H}_B = \mathcal{H}_A$, let $\{|i\rangle\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}_A$ (and of $\mathcal{H}_B$), and set $|\Omega\rangle = \sum_i |i\rangle|i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ (note that $|\Omega\rangle$ is not normalized). Show that

$$|\varphi\rangle = (\sqrt{\rho_A} \otimes \mathbb{I}_B)|\Omega\rangle$$

is a purification of $\rho_A$, i.e., $|\varphi\rangle \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\mathrm{tr}_B(|\varphi\rangle\langle\varphi|) = \rho_A$.

**Exercise 2.2** Use the Schmidt decomposition to show that for every $|\varphi\rangle \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the two reduced density matrices $\rho_A = \mathrm{tr}_B(|\varphi\rangle\langle\varphi|)$ and $\rho_B = \mathrm{tr}_A(|\varphi\rangle\langle\varphi|)$ have the same non-zero eigenvalues (with the same multiplicities).

**Exercise 2.3** Show that $\langle A|B\rangle = \mathrm{tr}(A^\dagger B)$ and $\left|AXB^T\right\rangle = (A \otimes B)|X\rangle$ for any $A, B, X \in \mathcal{L}(\mathcal{H})$, as claimed in Section 0.7.
*Hint:* It is sufficient to show the claims for (some of) the operators being of the form $|i\rangle\langle j|$. Note that, for simplicity, we consider $\mathcal{H} = \mathcal{H}'$ here, and so we have *one* fixed basis $\{|i\rangle\}_{i \in I}$ here.

**Exercise 2.4** Show that for any CPTP map $\mathfrak{T} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{A'})$, there exists a Kraus representation $\mathfrak{T}(R) = \sum_\ell T_\ell R T_\ell^\dagger$ with mutually *orthogonal* Kraus operators with respect to the Hilbert-Schmidt inner product, i.e. $\mathrm{tr}\big(T_\ell^\dagger T_k\big) = 0$ for all $\ell \neq k$.
*Hint:* Use how the Kraus operators $T_\ell$ are constructed in the proof of Theorem 6.6.

**Exercise 2.5** We consider yet another representation of a CPTP map: using the notation from Section 0.6 for fixed orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, any CPTP map $\mathfrak{T} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{A'})$ induces an operator $M(\mathfrak{T}) \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_A, \mathcal{H}_{A'} \otimes \mathcal{H}_{A'})$ that maps

$$M(\mathfrak{T}) : |R\rangle \mapsto |\mathfrak{T}(R)\rangle\,.$$

$M(\mathfrak{T})$ is called the *matrix representation* of $\mathfrak{T}$. Show that if $\mathfrak{T}(R) = \sum_\ell T_\ell R T_\ell^\dagger$ is the Kraus representation of $\mathfrak{T}$, then $M(\mathfrak{T}) = \sum_\ell T_\ell \otimes \bar{T}_\ell$.

[1] Existence and uniqueness of $\sqrt{\rho_A}$ follows immediately from the respective spectral decompositions of $\rho_A$ and its positive-semidefinite square root.