Exercise Set 3

Exercise 3.1 Consider the so-called *Pauli-X* operator

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{C}^2)$$

and compute the operator $e^{-i\theta X} \in \mathcal{L}(\mathbb{C}^2)$, defined according to Definition 0.1 by applying the function $x \mapsto e^{-i\theta x} = \cos(\theta x) - i \sin(\theta x)$ to the eigenvalues in the spectral decomposition of X.

Exercise 3.2 Show that $(L \otimes R)^p = L^p \otimes R^p$ holds for any $L \in \mathcal{P}(\mathcal{H})$, $R \in \mathcal{P}(\mathcal{H}')$, and $p \in \mathbb{R}$. Use this to show that $|L \otimes R| = |L| \otimes |R|$ and $\|L \otimes R\|_p = \|L\|_p \, \|R\|_p$ holds for any $L \in \mathcal{L}(\mathcal{H})$, $R \in \mathcal{L}(\mathcal{H}')$, and $0 \neq p \in \mathbb{R}$.

Exercise 3.3 Show that for any two density matrices $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, there exists an orthonormal basis $\{|i\rangle\}_{i \in I}$ of $\mathcal{H}$ such that the respective probability distributions $P, Q : I \to [0,1]$, given by $P(i) = \operatorname{tr}(|i\rangle\langle i|\rho)$ and $Q(i) = \operatorname{tr}(|i\rangle\langle i|\sigma)$, satisfy $\delta(P,Q) = \delta(\rho,\sigma)$.

Exercise 3.4 Let $A, B \in \mathcal{L}(\mathcal{H})$ be Hermitian. Show that if A and B have orthogonal supports, i.e., $\operatorname{supp}(A) \perp \operatorname{supp}(B)$, then $\|A + B\|_1 = \|A\|_1 + \|B\|_1$. Conclude Lemma 7.8.

Exercise 3.5 Show that for any random variables X and Y with joint distribution P_{XY} the following inequalities hold, where $\mathcal{Y}$ is the range of Y.

$$\mathrm{H}_\infty(XY) \geq \mathrm{H}_\infty(X)\,, \quad \mathrm{H}_\infty(X|Y) \leq \mathrm{H}_\infty(X) \quad \text{and} \quad \mathrm{H}_\infty(X|Y) \geq \mathrm{H}_\infty(XY) - \log|\mathcal{Y}|\,.$$