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# Exercise Set 4

**Exercise 4.1** Show that $\mathrm{D}_\alpha(\rho,\rho) = 0$ for any $\rho \in \mathcal{D}(\mathcal{H})$. With the help of the previous exercise set, show that $\mathrm{D}_\alpha(\rho \otimes \rho', \sigma \otimes \sigma') = \mathrm{D}_\alpha(\rho,\sigma) + \mathrm{D}_\alpha(\rho',\sigma')$ for all $\rho,\rho' \in \mathcal{D}(\mathcal{H})$ and $\sigma,\sigma' \in \mathcal{P}(\mathcal{H})$. Finally, using the fact that $\mathrm{D}_\alpha(\rho,\sigma) \geq 0$ for all $\rho,\sigma \in \mathcal{D}(\mathcal{H})$ (which we will later see to hold), show that $\mathrm{H}_\alpha(A|E) = \mathrm{H}_\alpha(A)$ in case of a product state $\rho_{AE} = \rho_A \otimes \rho_E$.

**Exercise 4.2** Recall the fidelity

$$F(\rho,\sigma) := \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_{tr}$$

for $\rho,\sigma \in \mathcal{D}(\mathcal{H})$. Show that it is symmetric, i.e., $F(\rho,\sigma) = F(\sigma,\rho)$. *Hint:* Use Lemma 0.4. Simplify $F(\rho,\sigma)$ in case $\sigma = |\psi\rangle\langle\psi|$ is pure. Show that $\mathrm{D}_{1/2}(\rho\|\sigma) = -2\log F(\rho,\sigma)$.

**Exercise 4.3** For arbitrary $\rho,\sigma \in \mathcal{D}(\mathcal{H})$ and $0 \leq \lambda \in \mathbb{C}$, how does $\mathrm{D}_\alpha(\rho\|\lambda\sigma)$ relate to $\mathrm{D}_\alpha(\rho\|\sigma)$? For $\rho_{AE} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_A)$, conclude that

$$\mathrm{H}_\alpha(A|E) = n - \min_{\sigma_E} \mathrm{D}_\alpha(\rho_{AE} \,\|\, \mu_A \otimes \sigma_E)\,,$$

where min is over all $\sigma_E \in \mathcal{D}(\mathcal{H}_E)$, and $\mu_A := \mathbb{I}_A/N \in \mathcal{D}(\mathcal{H}_A)$ with $N := \dim(\mathcal{H}_A)$ and $n := \log(N)$. Thus, $\mathrm{H}_\alpha(A|E)$ can be understood as $n$, the maximal possible entropy, achieved by a state of the form $\mu_A \otimes \sigma_E$, minus how far away $\rho_{AE}$ is from such a state.

**Exercise 4.4** Let $V, V'$ be vector spaces over $\mathbb{R}$ and $L$ a linear map $L : V' \to V$. Let $C \subseteq V$ be a convex set (i.e., $x,y \in C, 0 \leq p \leq 1 \Rightarrow px + (1-p)y \in C$) and $f$ a convex function on $C$. Show that the set $C' := L^{-1}(C) = \{x' \,|\, L(x') \in C\}$ and the function $f' := f \circ L$ with domain $C'$ are convex as well.

**Exercise 4.5** Let $\mathcal{X}$ and $\mathcal{Y}$ be convex sets, let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function, and let $g : \mathcal{X} \to \mathbb{R}$ be given by $g(x) = \max_y f(x,y)$, where the max is promised to exist (for any $x$). Show that:

1. if $f(\cdot, y)$ is *convex* (as a function of $x$) for every $y \in \mathcal{Y}$ then $g$ is convex, and
2. if $f$ is *jointly concave* then $g$ is concave.