Solutions to Exercise Set 1

Solution 1.1 By acting with $\langle\varphi|$ from the left and with $|\varphi\rangle$ from the right, and then taking absolute values, we see that

$$1 = |\langle\varphi|\varphi\rangle\langle\varphi|\varphi\rangle| = \left|\sum_{i=1}^{n} \varepsilon_i \langle\varphi|\varphi_i\rangle\langle\varphi_i|\varphi\rangle\right| \leq \sum_{i=1}^{n} \varepsilon_i |\langle\varphi|\varphi_i\rangle\langle\varphi_i|\varphi\rangle| \leq \sum_{i=1}^{n} \varepsilon_i = 1$$

where the first inequality is triangle inequality, and the second is Cauchy-Schwarz. But since the inequalities must be equalities, we have that every $|\varphi_i\rangle = \omega_i|\varphi\rangle$ for a scalar $\omega_i \in \mathbb{C}$ with $|\omega_i| = 1$, and thus $|\varphi_i\rangle\langle\varphi_i| = \omega\bar{\omega}|\varphi\rangle\langle\varphi| = |\varphi\rangle\langle\varphi|$.

Solution 1.2 Consider the spectral decomposition $\rho = \sum_i \lambda_i |i\rangle\langle i|$. Note that $\lambda_i = \langle i|\rho|i\rangle \geq 0$ (because $\rho \geq 0$) and $\sum_i \lambda_i = \operatorname{tr}(\rho) = 1$. Furthermore,

$$\operatorname{tr}(\rho^2) = \sum_{i,j} \lambda_i\lambda_j \operatorname{tr}(|i\rangle\langle i|j\rangle\langle j|) = \sum_{i,j} \lambda_i\lambda_j \operatorname{tr}(|i\rangle\langle i||j\rangle\langle j|) = \sum_i \lambda_i^2 \operatorname{tr}(|i\rangle\langle i|) = \sum_i \lambda_i^2 ,$$

where the right-hand side is obviously non-negative, and it is upper bounded by $(\sum_i \lambda_i)^2 = 1$, with equality if and only if $\lambda_i = 0$ for all but one i (again by Proposition 0.1), i.e., ρ is pure.

Solution 1.3 Consider the spectral decomposition $\rho = \sum_i \lambda_i |i\rangle\langle i|$ of ρ. Then,

$$\operatorname{tr}(\rho\,\sigma) = \sum_i \lambda_i \operatorname{tr}(|i\rangle\langle i|\sigma) = \sum_i \lambda_i \, \langle i|\sigma|i\rangle \, .$$

The lower bound follows because $\lambda_i \geq 0$ and $\sigma \geq 0$, and the upper bound follows from $\langle i|\sigma|i\rangle \leq 1$ and $\sum_i \lambda_i = 1$.

Solution 1.4 Let $\{|i\rangle\}_{i\in I}$ be an eigenbasis of $\rho - \sigma$, i.e., consist of the vectors from the spectral decomposition $\rho - \sigma = \sum_i \lambda_i |i\rangle\langle i|$. Since $\rho \neq \sigma$, there exists j with $\lambda_j \neq 0$. For that j, we have

$$\operatorname{tr}(|j\rangle\langle j|\rho) - \operatorname{tr}(|j\rangle\langle j|\sigma) = \operatorname{tr}\big(|j\rangle\langle j|(\rho - \sigma)\big) = \sum_i \lambda_i \operatorname{tr}\big(|j\rangle\langle j||i\rangle\langle i|\big) = \sum_i \lambda_i \langle i|j\rangle\langle j|i\rangle = \lambda_j \neq 0 \, .$$

Solution 1.5 Fix an orthonormal basis $\{|e_i\rangle\}_{i\in I}$, and consider arbitrary fixed indices $i, j \in I$. For $|\varphi\rangle = |e_i\rangle + |e_j\rangle$ and $|\varphi'\rangle = |e_i\rangle + i|e_j\rangle$, we see then that $0 = \langle\varphi|A|\varphi\rangle = \langle e_i|A|e_j\rangle + \langle e_j|A|e_i\rangle$ and $0 = \langle\varphi'|A|\varphi'\rangle = i\langle e_i|A|e_j\rangle - i\langle e_j|A|e_i\rangle$, where for the latter we used that $\langle\varphi'| = \langle e_i| - i\langle e_j|$. It now follows that $\langle e_i|A|e_j\rangle = 0$. Since $i \in I$ was arbitrary, we get that $A|e_j\rangle = 0$; indeed, only the 0-vector is orthogonal to a basis. Finally, since $j \in I$ was arbitrary, we then get $A = 0$.