Solutions to Exercise Set 2

Solution 2.1 Note that $|\varphi\rangle = \sum_i \sqrt{\rho_A}|i\rangle \otimes |i\rangle$ and $\sqrt{\rho_A}^\dagger = \sqrt{\rho_A}$, so that

$$|\varphi\rangle\langle\varphi| = \sum_{i,j} \sqrt{\rho_A}|i\rangle\langle j|\sqrt{\rho_A} \otimes |i\rangle\langle j|$$

and

$$\mathrm{tr}_B(|\varphi\rangle\langle\varphi|) = \sum_{i,j} \sqrt{\rho_A}|i\rangle\langle j|\sqrt{\rho_A}\,\langle i|j\rangle = \sum_i \sqrt{\rho_A}|i\rangle\langle i|\sqrt{\rho_A} = \rho_A\,.$$

That $|\varphi\rangle \in \mathcal{S}(\mathcal{H})$ follows directly from $\langle\varphi|\varphi\rangle = \mathrm{tr}(|\varphi\rangle\langle\varphi|) = \mathrm{tr}\big(\mathrm{tr}_B(|\varphi\rangle\langle\varphi|)\big) = \mathrm{tr}(\rho_A) = 1$.

Solution 2.2 By the Schmidt decomposition, we can write $|\varphi\rangle$ as

$$|\varphi\rangle = \sum_{i=1}^{r} \mu_i |e_i\rangle|f_i\rangle\,,$$

where $\{|e_1\rangle, \dots, |e_{d_A}\rangle\}$ and $\{|f_1\rangle, \dots, |f_{d_B}\rangle\}$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, $r \le d_A, d_B$, and $\mu_1, \dots \mu_r > 0$. Therefore, we can write

$$\rho_A = \mathrm{tr}_B(|\varphi\rangle\langle\varphi|) = \sum_{i=1}^{r} \mu_i^2 |e_i\rangle\langle e_i| \qquad \text{and} \qquad \rho_B = \mathrm{tr}_A(|\varphi\rangle\langle\varphi|) = \sum_{i=1}^{r} \mu_i^2 |f_i\rangle\langle f_i|$$

and thus have the respective spectral decompositions of ρ_A and ρ_B, and we observe that the non-zero eigenvalues of ρ_A as well as of ρ_B are given by the μ_i^2's, and thus are identical.

Solution 2.3 Let $\{|i\rangle\}_{i\in I}$ be the fixed orthonormal basis considered for the notation of Section 0.6. For $A = |i\rangle\langle j|$ and $B = |k\rangle\langle\ell|$ with $i,j,k,\ell \in I$ we have $|A\rangle = |i\rangle|j\rangle$ and $B = |k\rangle|\ell\rangle$, and thus

$$\langle A|B\rangle = (\langle i| \otimes \langle j|)(|k\rangle \otimes |\ell\rangle) = \langle i|k\rangle\langle j|\ell\rangle\,,$$

while

$$\mathrm{tr}(A^\dagger B) = \mathrm{tr}(|j\rangle\langle i||k\rangle\langle\ell|) = \langle i|k\rangle\,\mathrm{tr}(|j\rangle\langle\ell|) = \langle i|k\rangle\langle\ell|j\rangle\,.$$

Since $\langle j|\ell\rangle$ is 0 or 1, and thus $\langle j|\ell\rangle = \langle\ell|j\rangle$, the first claim holds for the considered A and B. For general A and B, the claim follows from linearity and the fact that the $|i\rangle\langle j|$'s form a basis.

As for the second claim, for $B = |i\rangle\langle j|$ and $X = |k\rangle\langle\ell|$ with $i,j,k,\ell \in I$ and for arbitrary A, we have

$$AXB^T = A|k\rangle\langle\ell||j\rangle\langle i| = A|k\rangle\langle\ell|j\rangle\langle i|\,,$$

and therefore, noting that (by linearity) the ket-vector representation maps $|\varphi\rangle\langle i| \mapsto |\varphi\rangle|i\rangle$ for *any* $|\varphi\rangle \in \mathcal{H}$, we have

$$\big|AXB^T\big\rangle = A|k\rangle \otimes |i\rangle\langle\ell|j\rangle = A|k\rangle \otimes |i\rangle\langle j|\ell\rangle = (A \otimes |i\rangle\langle j|)|k\rangle|\ell\rangle = (A \otimes B)|X\rangle$$

Again, this implies the claim for an arbitrary B and X by linearity.

Solution 2.4 The proof of Theorem 6.6 shows that the Kraus operators T_ℓ can be chosen so that $|T_\ell\rangle = \sqrt{\lambda}|e_\ell\rangle$, where $\sum_\ell \lambda_\ell |e_\ell\rangle\langle e_\ell|$ is the spectral decomposition of the Choi-Jamiołkowski representation $J(\mathfrak{T})$ of $\mathfrak{T}$. In particular, the ket vectors $|T_\ell\rangle$ are mutually orthogonal and thus $\mathrm{tr}\big(T_\ell^\dagger T_k\big) = \langle T_\ell|T_k\rangle = 0$ for all $\ell \ne k$.

Solution 2.5 Using properties as discussed in Section 0.7, we see that

$$|\mathfrak{T}(R)\rangle = \big|\textstyle\sum_\ell T_\ell R T_\ell^\dagger\big\rangle = \displaystyle\sum_\ell \big|T_\ell R T_\ell^\dagger\big\rangle = \sum_\ell \big|T_\ell R \bar{T}_\ell^T\big\rangle = \sum_\ell T_\ell \otimes \bar{T}_\ell\,|R\rangle\,,$$

which proves the claim.