Solutions to Exercise Set 3

Solution 3.1 A straightforward calculation shows that X has the eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

with respective eigenvalues ± 1. Thus, X has spectral decomposition $X = |+\rangle\langle+| - |-\rangle\langle-|$, and therefore, recalling that $|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{I}$, we obtain

$$\begin{aligned} e^{-i\theta X} &= e^{-i\theta}|+\rangle\langle+| + e^{i\theta}|-\rangle\langle-| \\ &= \cos(\theta)\big(|+\rangle\langle+| + |-\rangle\langle-|\big) - i\sin(\theta)\big(|+\rangle\langle+| - |-\rangle\langle-|\big) \\ &= \cos(\theta)\mathbb{I} - i\sin(\theta)X \end{aligned}$$

which, as matrix, equals

$$e^{-i\theta X} = \begin{bmatrix}\cos(\theta) & -i\sin(\theta)\\ -i\sin(\theta) & \cos(\theta)\end{bmatrix}.$$

Solution 3.2 First, we observe that for $L \in \mathcal{P}(\mathcal{H})$ and $R \in \mathcal{P}(\mathcal{H}')$ with respective spectral decompositions $L = \sum_i \lambda_i |e_i\rangle\langle e_i|$ and $R = \sum_j \mu_j |f_j\rangle\langle f_j|$ with $\lambda_i, \mu_j > 0$, it holds that

$$L \otimes R = \sum_{ij} \lambda_i \mu_j |e_i\rangle|f_j\rangle\langle e_i|\langle f_j|$$

is the spectral decomposition of $L \otimes R$, and therefore

$$(L\otimes R)^p = \sum_{ij} \lambda_i^p \mu_j^p |e_i\rangle|f_j\rangle\langle e_i|\langle f_j| = \sum_i \lambda_i^p |e_i\rangle\langle e_i| \otimes \sum_j \mu_j^p |f_j\rangle\langle f_j| = L^p \otimes R^p$$

for any power $p \in \mathbb{R}$. For $L \in \mathcal{L}(\mathcal{H})$ and $R \in \mathcal{L}(\mathcal{H}')$, this then implies that

$$|L\otimes R| = \sqrt{(L\otimes R)^\dagger (L \otimes R)} = \sqrt{L^\dagger L \otimes R^\dagger R} = \sqrt{|L|^2 \otimes |R|^2} = \sqrt{(|L|\otimes|R|)^2} = |L| \otimes |R|,$$

and thus that

$$\|L\otimes R\|_p^p = \mathrm{tr}\big(|L\otimes R|^p\big) = \mathrm{tr}\big((|L|\otimes|R|)^p\big) = \mathrm{tr}\big(|L|^p \otimes |R|^p\big) = \mathrm{tr}\big(|L|^p\big)\mathrm{tr}\big(|R|^p\big) = \|L\|_p^p\,\|R\|_p^p.$$

The claim follows by taking p-th roots.

Solution 3.3 Let $\{|i\rangle\}_{i\in I}$ be the eigenbasis of $\rho - \sigma$, i.e., consider the spectral decomposition $\rho - \sigma = \sum_i \lambda_i |i\rangle\langle i|$ of $\rho - \sigma$. Then,

$$\delta(P,Q) = \frac{1}{2}\sum_j \big|\mathrm{tr}(|j\rangle\langle j|\rho) - \mathrm{tr}(|j\rangle\langle j|\sigma)\big| = \frac{1}{2}\sum_j \big|\mathrm{tr}(|j\rangle\langle j|(\rho-\sigma))\big| = \frac{1}{2}\sum_j |\lambda_j| = \delta(\rho,\sigma),$$

which was to be shown.

Solution 3.4 Let

$$A = \sum_i \lambda_i |e_i\rangle\langle e_i| \qquad \text{and} \qquad B = \sum_j \mu_j |f_j\rangle\langle f_j|$$

be the respective spectral decompositions of A and B, where we restrict i and j to those with $\lambda_i \neq 0$ and $\mu_j \neq 0$, respectively. Then, as discussed in Sect. 0.3, for any such i with $\lambda_i \neq 0$

and j with $\mu_j \neq 0$, it holds that $|e_i\rangle \in \text{supp}(A)$ and $|f_j\rangle \in \text{supp}(B)$, and so, by assumption $\langle e_i|f_j\rangle = 0$ for all those i and j's. Therefore,

$$A + B = \sum_i \lambda_i |\varepsilon_i\rangle\langle\varepsilon_i| + \sum_j \mu_j |f_j\rangle\langle f_j|$$

forms the spectral decomposition of $A + B$, and so

$$\|A + B\|_1 = \sum_i |\lambda_i| + \sum_j |\mu_j| = \|A\|_1 + \|B\|_1 \,.$$

Lemma 7.8 follows by observing that $\rho_{XE} - \rho'_{XE} = \sum_x P_X(x)|x\rangle\langle x| \otimes (\rho_E^x - \rho_E'^x)$, and that the $|x\rangle\langle x| \otimes (\rho_E^x - \rho_E'^x)$'s have pairwise orthogonal supports. Indeed, the support of $|x\rangle\langle x| \otimes (\rho_E^x - \rho_E'^x)$ is contained in $\text{span}(|x\rangle) \otimes \mathcal{H}_E$.

Solution 3.5 For the first inequality, we see that

$$\text{Guess}(X) = \max_x P_X(x) = \max_x \sum_y P_{XY}(x,y) \geq \max_x \max_y P_{XY}(x,y) = \text{Guess}(XY) \,,$$

which implies the claim. For the second, we observe that

$$\begin{aligned}\text{Guess}(X|Y) &= \sum_y P_Y(y) \max_x P_{X|Y}(x|y) = \sum_y \max_x P_{XY}(x,y) \\ &\geq \max_x \sum_y P_{XY}(x,y) = \max_x P_X(x) = \text{Guess}(X) \,.\end{aligned}$$

Finally, recycling part of above, we get

$$\text{Guess}(X|Y) = \sum_y \max_x P_{XY}(x,y) \leq \sum_y \max_{x,y'} P_{XY}(x,y') = |\mathcal{Y}|\text{Guess}(XY)$$

which again gives us the claimed inequality by taking $-\log$ on both sides.